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Spatial dynamics of a nonlocal delayed unstirred chemostat model with periodic input

Wu-Jun Pu*, Danhua Jiang†,§ Ya Wang† and Zhanbing Bai‡

*Department of Mathematics, Longnan Teachers College
Longnan, Gansu 742500, P. R. China

†School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu 730000, P. R. China

‡College of Mathematics and System Science
Shandong University of Science and Technology
Qingdao, Shandong 266590, P. R. China
§jiangdh15@lzu.edu.cn

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In this paper, a nonlocal delayed chemostat model of a single species feeding on a periodically varying input nutrient is proposed. By the theory of semigroup, the existence and uniqueness of solution of the system are obtained. Furthermore, we investigate a threshold result on the global dynamics, and the uniform persistence of the system is established.

Keywords: Chemostat; nonlocal delay; uniform persistence; periodic solution.

Mathematics Subject Classification 2010: 35K55, 35K57, 37N25, 92C17

1. Introduction

A chemostat is an apparatus used to study the growth of microorganisms in a continuous cultured environment in a laboratory. In ecology, the chemostat is a model of a simple lake, but in chemical engineering, it also serves as a laboratory model of a bio-reactor used to manufacture products with genetically altered organisms. In more complicated situations, it is used as the starting point for models of waste water treatment [22] or of the mammalian large intestine [5]. Hence, chemostat models have attracted much attention of both biologists and mathematicians. Analytic work on the chemostat models can be found in [4, 25, 29] and references therein. At present, the research on the chemostat model has been generalized in several directions, including spatial effects in a continuous way (see, e.g., [1, 9, 10]),

§Corresponding author.

the effect of inhibitor or toxin (see e.g. [18, 19, 34]), crowding effects (see e.g. [11]), periodically varying input nutrient concentration (see, e.g. [26, 30]), delays (see, e.g. [20, 21, 31, 32, 36, 37]), etc.

In order to study habitats, such as riverine reservoirs constructed by damming a river, we first review the flowing-reactor model of the organism which consumes the single nutrient in a riverine reservoir occupying a simple channel. Assume that the channel has constant length L. A flow of medium in the channel with velocity ν in the direction of increasing x brings fresh nutrient at concentration R^0 into the reactor at the upstream end ($x = 0$) and carries medium, unutilized nutrient and organism out of the reactor at the downstream ($x = L$). Nutrient and organism are assumed to diffuse throughout the vessel with the same diffusivity δ. The equations describing the nutrient concentration R and organism concentration u are then given below (see, e.g. [13] or [23, Chap. 8])

$$\begin{cases} \dfrac{\partial R}{\partial t} = \delta\dfrac{\partial^2 R}{\partial x^2} - \nu\dfrac{\partial R}{\partial x} - qf(R)u, & x \in (0, L),\ t > 0, \\ \dfrac{\partial u}{\partial t} = \delta\dfrac{\partial^2 u}{\partial x^2} - \nu\dfrac{\partial u}{\partial x} - \mu_0 u + f(R)u, & x \in (0, L),\ t > 0, \end{cases} \tag{1.1}$$

with boundary conditions

$$\begin{cases} \nu R(t, 0) - \delta\dfrac{\partial R}{\partial x}(t, 0) = \nu R^{(0)}(t), & \dfrac{\partial R}{\partial x}(t, L) = 0, \\ \nu u(t, 0) - \delta\dfrac{\partial u}{\partial x}(t, 0) = 0, & \dfrac{\partial u}{\partial x}(t, L) = 0, \end{cases} \tag{1.2}$$

and initial conditions

$$R(0, x) = R_0(x) \geq 0, \quad u(0, x) = u_0(x) \geq 0,\ \ x \in (0, L), \tag{1.3}$$

where q is the constant nutrient quota for species, μ_0 is the death rate of species. The nonlinear function $f(R)$ describes the nutrient uptake rate and the growth rate of the organism u at nutrient concentration R. We assume $f(R)$ satisfies

$$f(0) = 0, \quad f'(R) > 0, \quad \forall\, R \geq 0, \quad f(\cdot) \in C^2(0, +\infty). \tag{1.4}$$

An usual example is the Monod function

$$f(R) = \frac{\mu R}{K + R}, \quad \forall\, R \geq 0,$$

where the constants $\mu > 0$ and $K > 0$ are, respectively, the maximum growth rate and half-saturation coefficient.

If we consider the periodic time dependence in the nutrient concentration to account for seasonal or daily changes, then the model is more realistic. Based on this reason, we assume that the nutrient concentration in the medium is maintained at the periodically varying concentration $R^{(0)}(t + \omega) = R^{(0)}(t)$ at the up stream end of the channel ($x = 0$), where $\omega > 0$ is the periodic.

Furthermore, it is well known that time delays in ecological systems can have a considerable influence on the qualitative behavior of these systems. It is the purpose of this paper to propose and analyze a more realistic chemostat model. According to Droop [3], only the internal nutrient is immediately available for cell growth, and passage of nutrient from outside to inside the cells introduces inevitable time delays. Thus, the assumption that the external nutrient supply is instantaneously converted to biomass is a broad oversimplification and should (at least in part) account for the inadequacies of the Monod model in a chemostat. Now, we derive a nonlocal delayed reaction-diffusion-advection chemostat model.

In the organism equation, the delay is often caused by the conversion of consumed nutrient into organism biomass. In order to incorporate the movements of organism during the assimilation process, we consider an age-structured model. Let $w(t, a, x)$ be the generalized organism biomass (organism biomass plus nutrient ingested by the organism that has not yet been assimilated) of class age a, with a being the time since ingestion. If $p(a)$ denotes the probability that the absorbed nutrients are completely translated into their own biomass at age $a \geq 0$, and conversion occurs after a fixed delay τ, then $p(a)$ is a step function, i.e., $p = 0$ on $[0, \tau)$ and $p = 1$ on (τ, ∞). Then the microorganism biomass at time $t \geq 0$ and location x can be expressed as

$$u(t, x) = \int_0^\infty w(t, a, x)p(a)da = \int_\tau^\infty w(t, a, x)da.$$

Moreover, the generalized biomass density $w(t, a, x)$ satisfies the following partial differential equation with nonlocal boundary condition:

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right) w(t, a, x) = \delta \dfrac{\partial^2 w}{\partial x^2} - \nu \dfrac{\partial w}{\partial x} - \mu_0 w, & t > 0,\ a \in (0, \tau),\ x \in (0, L), \\ w(t, 0, x) = f(R(t, x))u(t, x), & t \geq -\tau,\ x \in (0, L). \end{cases} \tag{1.5}$$

The parameters in this system have the same meaning as in (1.1).

Similarly, as in [24, 28], we integrate along characteristics to reduce (1.5) to one equation with nonlocal terms. Let $\phi(r, a, x) = w(a + r, a, x)$, where r is regarded as a parameter. It follows that

$$\begin{cases} \dfrac{\partial \phi(r, a, x)}{\partial a} = \delta \dfrac{\partial^2 \phi(r, a, x)}{\partial x^2} - \nu \dfrac{\partial \phi(r, a, x)}{\partial x} - \mu_0 \phi(r, a, x), \\ \phi(r, 0, x) = f(R(r, x))u(r, x). \end{cases} \tag{1.6}$$

Integrating this equation on $(0, L)$, we have

$$\phi(r, a, x) = \int_0^L \Gamma(a, x, y)f(R(r, y))u(r, y)dy,$$

where Γ is the appropriate Greens function or fundamental solution associated with $\delta\frac{\partial^2}{\partial x^2} - \nu\frac{\partial}{\partial x} - \mu_0$ and possibly boundary conditions. Returning to w, one has

$$w(t,a,x) = \phi(t-a,a,x) = \int_0^L \Gamma(a,x,y) f(R(t-a,y)) u(t-a,y)dy.$$

This yields the following integral equation for the microorganisms biomass u:

$$u(t,x) = \int_\tau^\infty w(t,a,x)da = \int_\tau^\infty \int_0^L \Gamma(a,x,y) f(R(t-a,y)) u(t-a,y)dyda.$$

Set $t-a=s$. Then

$$u(t,x) = \int_{-\infty}^{t-\tau} \int_0^L \Gamma(t-s,x,y) f(R(s,y)) u(s,y)dyds.$$

Differentiating the above equation, we obtain

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \delta\frac{\partial^2 u(t,x)}{\partial x^2} - \nu\frac{\partial u(t,x)}{\partial x} - \mu_0 u(t,x)\\ &\quad + \int_0^L \Gamma(\tau,x,y) f(R(t-\tau,y)) u(t-\tau,y)dy.\end{aligned}$$

Therefore, we obtain the following time-delayed reaction-diffusion-advection chemostat model with periodically varying input nutrient concentration:

$$\begin{cases} \dfrac{\partial R}{\partial t} = \delta\dfrac{\partial^2 R}{\partial x^2} - \nu\dfrac{\partial R}{\partial x} - qf(R(t,x))u, & t>0,\ \ x\in(0,L),\\ \dfrac{\partial u}{\partial t} = \delta\dfrac{\partial^2 u}{\partial x^2} - \nu\dfrac{\partial u}{\partial x} - \mu_0 u & \\ \qquad\qquad + \displaystyle\int_0^L \Gamma(\tau,x,y) f(R(t-\tau,y)) u(t-\tau,y)dy, & t>0,\ \ x\in(0,L),\\ \nu R(t,0) - \delta\dfrac{\partial R}{\partial x}(t,0) = \nu R^{(0)}(t), \quad \dfrac{\partial R}{\partial x}(t,L) = 0, & t>0,\\ \nu u(t,0) - \delta\dfrac{\partial u}{\partial x}(t,0) = 0, \quad \dfrac{\partial u}{\partial x}(t,L) = 0, & t>0,\\ R(s,x) = R_0(s,x) \ge 0, \quad u(s,x) = u_0(s,x) \ge 0, & s\in[-\tau,0],\ \ x\in(0,L), \end{cases} \tag{1.7}$$

where $R(t,x)$ and $u(t,x)$ denote the concentrations of the nutrient and the microorganism at time t and position x; the constants $\delta > 0$ and $\nu > 0$ are the diffusion coefficient and advection coefficient, respectively; $\tau > 0$ is the time delay. The inflow nutrient concentration $R^{(0)}(t)$ satisfies

(H) $R^{(0)}(\cdot) \in C^2((-\tau,\infty),\mathbb{R})$, $R^{(0)}(t) \ge, \not\equiv 0$ and $R^{(0)}(t+\omega) = R^{(0)}(t)$ for some $\omega > 0$.

The purpose of this paper is to study the global dynamics of system (1.7) and the rest of this paper is organized as follows. In Sec. 2, we first present some preliminaries, then investigate the existence and uniqueness of the global solution of system (1.7). In Sec. 3, we obtain a threshold-type result on the global dynamics of system (1.7). Finally, we give a brief discussion of this paper in Sec. 4.

2. Existence and Uniqueness of Solution

Let $\mathbb{X} = C([0, L], \mathbb{R}^2)$ be the Banach space with the usual supremum norm $\|\cdot\|_{\mathbb{X}}$. Then $\mathbb{X}^+ = C([0, L], \mathbb{R}^2_+)$ is the positive cone of $\mathbb{X}$. For $\tau \geq 0$, define $C_\tau = C([-\tau, 0], \mathbb{X})$ with the norm $\|\phi\| = \max_{\theta\in[-\tau,0]} \|\phi(\theta)\|_{\mathbb{X}}, \forall\phi \in C_\tau$. Then C_τ is a Banach space and $C_\tau^+ = C([-\tau, 0], \mathbb{X}^+)$ is the positive cone of C_τ. Denote the inclusion $\mathbb{X} \to C_\tau$ by $\mathbf{u} \to \hat{\mathbf{u}}$, $\hat{\mathbf{u}}(\theta) = \mathbf{u}$, $\theta \in [-\tau, 0]$. Given a function $\mathbf{u}(t) : [-\tau, \sigma) \to \mathbb{X}$ $(\sigma > 0)$, define $\mathbf{u}_t \in C_\tau$ by $\mathbf{u}_t(\theta) = \mathbf{u}(t + \theta)$, $\theta \in [-\tau, 0]$.

The idea is to view system (1.7) as the abstract ordinary differential equation in $\mathbb{X}^+$ and the so-called mild solutions can be obtained for any given initial data. Let $T(t)$ be the positive, non-expansive, analytic semigroup on $C([0, L], \mathbb{R})$ (see, e.g. [23, Chap. 7]) such that $z = T(t)z_0$ satisfies the linear initial value problem

$$\begin{cases} \dfrac{\partial z}{\partial t} = \delta\dfrac{\partial^2 z}{\partial x^2} - \nu\dfrac{\partial z}{\partial x}, & t > 0,\ \ 0 < x < L, \\[2ex] \nu z(t, 0) - \delta\dfrac{\partial z}{\partial x}(t, 0) = \dfrac{\partial z}{\partial x}(t, L) = 0, & t > 0, \\[2ex] z(0, x) = z_0(x), & 0 < x < L. \end{cases} \tag{2.1}$$

Let $V(t, s)(t > s)$ be the evolution operator on $C([0, L], \mathbb{R})$ (see, e.g. [7, Chap. II]) such that $v = V(t, s)v_0$ satisfies the linear system with nonhomogeneous, periodic boundary conditions, with start time s, given by

$$\begin{cases} \dfrac{\partial v}{\partial t} = \delta\dfrac{\partial^2 v}{\partial x^2} - \nu\dfrac{\partial v}{\partial x}, & t > s,\ \ 0 < x < L, \\[2ex] \nu v(t, 0) - \delta\dfrac{\partial v}{\partial x}(t, 0) = vR^{(0)}(t), \quad \dfrac{\partial v}{\partial x}(t, L) = 0, & t > s, \\[2ex] v(s, x) = v_0(x), & 0 < x < L. \end{cases} \tag{2.2}$$

Due to the time periodicity of the inhomogeneity in the boundary condition, $R^0(t + \omega) = R^0(t)$, it follows from [2, Lemma 6.1] that

$$V(t + \omega, s + \omega) = V(t, s), \quad \forall\, t > s.$$

Define $\mathbf{F} = (F_1, F_2) : C_\tau^+ \to C_\tau^+$ by

$$F_1(\phi_1, \phi_2)(x) = -qf(\phi_1(0, x))\phi_2(0, x),$$

$$F_2(\phi_1, \phi_2)(x) = \int_0^L \Gamma(\tau, x, y)f(\phi_1(-\tau, y))\phi_2(-\tau, y)dy,$$

where $x \in [0, L]$, $\phi = (\phi_1, \phi_2) \in C_\tau^+$. Set $\mathbf{u} = (R, u)$, then

$$\begin{cases} R(t) = V(t,0)R_0 + \displaystyle\int_0^t T(t-s)F_1(\mathbf{u}(s))ds, \\ u(t) = e^{-\mu_0 t}T(t)u_0 + \displaystyle\int_0^t e^{-\mu_0(t-s)}T(t-s)F_2(\mathbf{u}(s))ds. \end{cases}$$

Therefore, system (1.7) can be expressed as

$$\mathbf{u}(t) = \mathbf{U}(t,0)\phi(0) + \int_0^t \mathbf{T}(t-s)\mathbf{F}(\mathbf{u}(s))ds, \quad t \geq 0, \;\; \phi \in C_\tau^+,$$

where

$$\mathbf{U}(t,s) = \begin{pmatrix} V(t,s) & 0 \\ 0 & e^{-\mu_0(t-s)}T(t-s) \end{pmatrix}$$

and

$$\mathbf{T}(t-s) = \begin{pmatrix} T(t-s) & 0 \\ 0 & e^{-\mu_0(t-s)}T(t-s) \end{pmatrix}.$$

To show the global existence of solutions of (1.7), we first consider the following differential equation:

$$\begin{cases} \dfrac{\partial \hat{R}(t,x)}{\partial t} = \delta \dfrac{\partial^2 \hat{R}}{\partial x^2} - \nu \dfrac{\partial \hat{R}}{\partial x}, \quad t > 0, \;\; 0 < x < L, \\ \nu \hat{R}(t,0) - \delta \dfrac{\partial \hat{R}}{\partial x}(t,0) = \nu R^{(0)}(t), \quad \dfrac{\partial \hat{R}}{\partial x}(t,L) = 0, \quad t > 0. \end{cases} \tag{2.3}$$

The following result is concerned with the dynamics of (2.3).

Lemma 2.1 ([26, Proposition 2.1]). *System* (2.3) *admits a unique positive ω-periodic solution $R^*(t,x)$, and for any $\hat{R}_0(x) \in C([0,L],\mathbb{R})$, the unique solution $\hat{R}(t,x)$ of* (2.3) *with $\hat{R}(0,x) = \hat{R}_0(x)$ satisfies*

$$\lim_{t\to\infty} (\hat{R}(t,x) - R^*(t,x)) = 0,$$

uniformly for $x \in [0, L]$.

The following result shows that solutions of system (1.7) exist globally on $[0, \infty)$.

Theorem 2.2. *For every initial data $\phi \in C_\tau^+$, system* (1.7) *has a unique solution $\mathbf{u}(t,\phi)$ on $[0,\infty)$ with $\mathbf{u}_0 = \phi$. Furthermore, system* (1.7) *generates an ω-periodic semiflow $\Phi_t := \mathbf{u}_t(\cdot) : C_\tau^+ \to C_\tau^+$, i.e. $\Phi_t(\phi)(s,x) = \mathbf{u}(t+s,x;\phi), \forall \phi \in C_\tau^+$, $t \geq 0, s \in [-\tau, 0], x \in [0, L]$, and Φ_t has a global compact attractor in C_τ^+.*

Proof. First, we show the local existence of the unique mild solution. Clearly, $\mathbf{F}$ is locally Lipschitz continuous. In view of (1.4), there exists $M > 0$ such that

$0 \le f(R) \le MR$, $\forall R \ge 0$. For any $\phi \in C_\tau^+$ and $h \ge 0$, we have

$$\phi(0,x) + h\mathbf{F}(\phi)(x) = \begin{pmatrix} \phi_1(0,x) + h[-qf(\phi_1(0,x))\phi_2(0,x)] \\ \phi_2(0,x) + h\displaystyle\int_0^L \Gamma(\tau,x,y)f(\phi_1(-\tau,y))\phi_2(-\tau,y)dy \end{pmatrix}$$

$$\ge \begin{pmatrix} \phi_1(0,x)[1 - hqM\phi_2(0,x)] \\ \phi_2(0,x) \end{pmatrix}, \quad t \ge 0, x \in (0,L).$$

The above inequality implies that $\phi(0,x) + h\mathbf{F}(\phi)(x) \in \mathbb{X}^+$ if h is sufficiently small. Therefore,

$$\lim_{h\to 0^+} \frac{1}{h}\mathrm{dist}(\phi(0,\cdot) + h\mathbf{F}(\phi), \mathbb{X}^+) = 0, \quad \forall\, \phi \in C_\tau^+. \tag{2.4}$$

By [17, Corollary 4], it then follows that for every $\phi \in C_\tau^+$, system (1.7) has a unique noncontinuable mild solution $\mathbf{u}(t,x;\phi)$ with $\mathbf{u}_0(\cdot,\cdot;\phi) = \phi$ and $\mathbf{u}(t,\cdot;\phi) \in \mathbb{X}^+$ for any t on its maximal interval of existence $[0,\sigma_\phi)$, where $\sigma_\phi \le \infty$. Moreover, by the analyticity of $\mathbf{U}(t,s), s,t \in \mathbb{R}, s < t$, $\mathbf{u}(t,x;\phi)$ is a classical solution of (1.7) when $t > \tau$.

Next, we use similar arguments to those in [27, Theorem 2.1] to prove the ultimate boundedness of solutions. Note that the first equation in (1.7) is dominated by (2.3), and it follows from Lemma 2.1 that (2.3) admits a unique positive ω-periodic solution $R^*(t,x)$ which is globally asymptotically stable in $C([0,L],\mathbb{R})$, the parabolic comparison theorem implies that $R(t,x)$ is bounded on $[0,\sigma_\phi)$. Then there is a constant $B_1 > 0$ such that for any $\phi \in C_\tau^+$, there exists a positive integer $l_1 = l_1(\phi) > 0$ satisfying $R(t,x;\phi) \le B_1$ for all $t \ge l_1\omega$ and $x \in [0,L]$.

For any given $\phi \in C_\tau^+$, let $(R(t,x),u(t,x)) := (R(t,\phi)(x),u(t,\phi)(x)), t \ge 0$, $x \in (0,L)$. Set

$$\bar{R}(t) = \int_0^L R(t,x)dx, \quad \bar{u}(t) = \int_0^L u(t,x)dx.$$

Integrating the first equation of (1.7) on $(0,L)$, by Greens formula, we obtain

$$\begin{aligned} \frac{d\bar{R}}{dt} &= \int_0^L \left(\delta\frac{\partial^2 R}{\partial x^2} - \nu\frac{\partial R}{\partial x}\right)dx - q\int_0^L f(R(t,x))u(t,x)dx \\ &= \left(\delta\frac{\partial R}{\partial x} - \nu R\right)(t,L) - \left(\delta\frac{\partial R}{\partial x} - \nu R\right)(t,0) - q\int_0^L f(R(t,x))u(t,x)dx \\ &= \nu R^{(0)}(t) - \nu R(t,L) - q\int_0^L f(R(t,x))u(t,x)dx \\ &\le \nu R^{(0)}(t) - q\int_0^L f(R(t,x))u(t,x)dx, \quad t > 0, \end{aligned}$$

i.e.

$$\int_0^L f(R(t,x))u(t,x)dx \le \frac{1}{q}\left[\nu R^{(0)}(t) - \frac{d\bar{R}}{dt}\right], \quad t > 0.$$

By the property of the fundamental solutions, there exists $k_0 > 0$ such that

$$\begin{aligned}\int_0^L \Gamma(\tau,x,y)f(R(t-\tau,y))u(t-\tau,y)dy &\le k_0 \int_0^L f(R(t-\tau,y))u(t-\tau,y)dy \\ &\le \frac{k_0}{q}\left[\nu R^{(0)}(t-\tau) - \frac{d\bar{R}(t-\tau)}{dt}\right].\end{aligned}$$

Integrating the second equation of (1.7) on $(0,L)$ yields

$$\begin{aligned}\frac{d\bar{u}}{dt} &= \int_0^L \left(\delta\frac{\partial^2 u}{\partial x^2} - \nu\frac{\partial u}{\partial x}\right)dx - \mu_0\int_0^L u(t,x)dx \\ &\quad + \int_0^L\int_0^L \Gamma(\tau,x,y)f(R(t-\tau,y))u(t-\tau,y)dydx \\ &= -\nu u(t,L) - \mu_0\bar{u}(t) + \int_0^L\int_0^L \Gamma(\tau,x,y)f(R(t-\tau,y))u(t-\tau,y)dydx, \\ &\le -\nu u(t,L) - \mu_0\bar{u}(t) + \frac{k_0}{q}\int_0^L\left[\nu R^{(0)}(t-\tau) - \frac{d\bar{R}(t-\tau)}{dt}\right]dx \\ &\le -\mu_0\bar{u}(t) + \frac{k_0}{q}\int_0^L\left[\nu R^{(0)}(t-\tau) - \frac{d\bar{R}(t-\tau)}{dt}\right]dx \\ &= -\mu_0\bar{u}(t) + \frac{k_0}{q}L\left[\nu R^{(0)}(t-\tau) - \frac{d\bar{R}(t-\tau)}{dt}\right] \\ &\le -\mu_0\bar{u}(t) + k_1 - k_2\frac{d\bar{R}(t-\tau)}{dt}, \quad t \ge l_1\omega + \tau,\end{aligned}$$

where $k_1 = k_0 L\nu \cdot \max_{t\in[0,\omega]}\{R^{(0)}(t-\tau)\}/q$ and $k_2 = k_0 L/q$.

On the other hand, since

$$e^{\mu_0 t}\frac{d\bar{u}(t)}{dt} = \frac{d[e^{\mu_0 t}\bar{u}(t)]}{dt} - \mu_0 e^{\mu_0 t}\bar{u}(t),$$

then

$$\frac{d[e^{\mu_0 t}\bar{u}(t)]}{dt} \le k_1 e^{\mu_0 t} - k_2 e^{\mu_0 t}\frac{d\bar{R}(t-\tau)}{dt}, \quad t \ge l_1\omega + \tau. \tag{2.5}$$

Integrating (2.5) by parts over $[l_1\omega + \tau, t]$, we can find a positive number k_3, independent of ϕ, and a positive number $k_4 = k_4(\phi)$, dependent on ϕ, such that

$$\bar{u}(t) \le k_4(\phi)e^{-\mu_0 t} + k_3, \quad t \ge l_1\omega + \tau.$$

Since $\Gamma(\tau,\cdot,\cdot)$ and $R(\cdot,\cdot)$ are bounded, it follows from the second equation in (1.7) that

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \delta\frac{\partial^2 u}{\partial x^2} - \nu\frac{\partial u}{\partial x} - \mu_0 u(t,x) + \int_0^L \Gamma(\tau,x,y) f(R(t-\tau,y))u(t-\tau,y)dy \\ &\le \delta\frac{\partial^2 u}{\partial x^2} - \nu\frac{\partial u}{\partial x} - \mu_0 u(t,x) + M_1\int_0^L u(t-\tau,y)dy \\ &= \delta\frac{\partial^2 u}{\partial x^2} - \nu\frac{\partial u}{\partial x} - \mu_0 u(t,x) + M_1\bar{u}(t-\tau),\end{aligned}$$

with some constant $M_1 > 0$. By the standard parabolic comparison theorem, there exists a positive number B_2, independent of the initial value ϕ, and $l_2 = l_2(\phi) > l_1(\phi)$ such that $u(t,x;\phi) \le B_2$ for any $t \ge l_2\omega + \tau$ and $x \in [0,L]$. Therefore, we have $\sigma_\phi = \infty$ for each $\phi \in C_\tau^+$.

Define the solution semiflow $\Phi_t = \mathbf{u}_t(\cdot) : C_\tau^+ \to C_\tau^+$, $t \ge 0$. It is easy to see that $\{\Phi_t\}_{t\ge 0}$ is an ω-periodic semiflow on C_τ^+. By the above arguments, we conclude that $\Phi_t : C_\tau^+ \to C_\tau^+$ is point dissipative. Moreover, $\Phi_t : C_\tau^+ \to C_\tau^+$ is compact for each $t > \tau$ by [33, Theorem 2.2.6]. Then, by [6, Theorem 3.4.8], $\Phi_t = \mathbf{u}_t(\cdot) : C_\tau^+ \to C_\tau^+, t \ge 0$ has a global compact attractor. □

3. Global Dynamics

In this section, we study the global dynamics of system (1.7). Note that system (1.7) reduces to (2.3) for $\hat{R} = R$ when $u \equiv 0$, it then follows from Lemma 2.1 that $(R^*(t,x),0)$ is an ω-periodic solution of system (1.7). Linearizing system (1.7) at $(R^*(t,x),0)$, we obtain the following linear system:

$$\begin{cases}\dfrac{\partial S}{\partial t} = \delta\dfrac{\partial^2 S}{\partial x^2} - \nu\dfrac{\partial S}{\partial x} - \mu_0 S \\ \qquad\qquad + \displaystyle\int_0^L \Gamma(\tau,x,y) f(R^*(t-\tau,y))S(t-\tau,y)dy, & t>0,\ x\in(0,L), \\ \nu S(t,0) - \delta\dfrac{\partial S}{\partial x}(t,0) = \nu R^{(0)}(t), \quad \dfrac{\partial S}{\partial x}(t,L) = 0, & t>0, \\ S(s,x) = \varphi(s,x), & s\in[-\tau,0],\ x\in(0,L).\end{cases} \tag{3.1}$$

Let $\mathbb{Y} = C([0,L],\mathbb{R})$ and $\mathbb{Y}^+ = C([0,L],\mathbb{R}^+)$. For $\tau \ge 0$, define $\mathcal{E} = C([-\tau,0],\mathbb{Y})$ with the norm $\|\varphi\| = \max_{\theta\in[-\tau,0]}\|\varphi(\theta)\|_{\mathbb{Y}}$, $\forall\varphi\in\mathcal{E}$, and $\mathcal{E}^+ = C([-\tau,0],\mathbb{Y}^+)$. Then $(\mathcal{E},\mathcal{E}^+)$ is a strongly ordered Banach space. By similar arguments as [12, Sec. 3], it follows that for any $\varphi\in\mathcal{E}^+$, system (3.1) has a unique mild solution $S(t,x;\varphi)$ with $S_0(\cdot,\cdot;\varphi) = \varphi$ and $S_t(\cdot,\cdot;\varphi)\in\mathcal{E}^+$ for all $t\ge 0$. Moreover, $S(t,x;\varphi)$ is a classic solution when $t>\tau$ and the comparison theorem holds for (3.1).

Define the Poincaré map of (3.1) $P : \mathcal{E} \to \mathcal{E}$ by $P(\varphi) = S_\omega(\varphi)$ for all $\varphi \in \mathcal{E}$, where $S_\omega(\varphi)(s,x) = S(\omega + s, x; \varphi)$ for all $(s,x) \in [-\tau, 0] \times [0, L]$, and $S(t,x;\varphi)$ is the solution of (3.1) with $S(s,x) = \varphi(s,x)$. By similar arguments to [12, Sec. 2], we can show that $S(t,x;\varphi) > 0$ for $t > \tau, x \in [0,L], \varphi \in \mathcal{E}^+$ with $\varphi \not\equiv 0$, and hence $S_t(\cdot,\cdot;\varphi)$ is strongly positive for $t > 2\tau$. Moreover, $S_t(\cdot,\cdot;\varphi)$ is compact on $\mathcal{E}^+$ for all $t > \tau$. Thus there exists an integer $n_0 > \frac{2\tau}{\omega}$, such that $P^{n_0} = S_{n_0\omega}$ is compact and strongly positive. Let $r_0 = r(P)$ be the spectral radius of P. It follows from [14, Lemma 3.1] that $r_0 > 0$ is a simple eigenvalue of P having a strong positive eigenvector $\bar{\phi} \in \mathrm{Int}(\mathcal{E}^+)$, and the modulus of any other eigenvalue is less than r_0.

Lemma 3.1. *Let $\lambda = -\frac{1}{\omega}\ln r_0$. Then there exists a positive ω-periodic function $v(t,x)$ such that $e^{-\lambda t}v(t,x)$ is a solution of* (3.1).

Proof. By the definitions of r_0 and $\bar{\phi}$, we have $P\bar{\phi} = r_0\bar{\phi}$. Let $S(t,x;\bar{\phi})$ be the solution of (3.1) with $S(s,x) = \bar{\phi}(s,x)$ for all $(s,x) \in [-\tau,0] \times (0,L)$. Since $\bar{\phi} \gg 0$, then $S(\cdot,\cdot;\bar{\phi}) \gg 0$. Let $\lambda = -\frac{1}{\omega}\ln r_0$ and $v(t,x) = e^{\lambda t}S(t,x;\bar{\phi})$ for all $t \geq -\tau, x \in (0,L)$. Then $r_0 = e^{-\lambda\omega}$ and $v(t,x) > 0$ for all $t \geq -\tau, x \in (0,L)$. Moreover,

$$\begin{aligned}
\frac{\partial v(t,x)}{\partial t} &= e^{\lambda t}\frac{\partial S(t,x;\bar{\phi})}{\partial t} + \lambda e^{\lambda t}S(t,x;\bar{\phi}) \\
&= e^{\lambda t}\left[\delta\frac{\partial^2 S(t,x;\bar{\phi})}{\partial x^2} - \nu\frac{\partial S(t,x;\bar{\phi})}{\partial x} - \mu_0 S(t,x;\bar{\phi})\right. \\
&\quad \left. + \int_0^L \Gamma(\tau,x,y)f(R^*(t-\tau,y))S(t-\tau,y;\bar{\phi})dy\right] + \lambda v(t,x) \\
&= \delta\frac{\partial^2 v(t,x)}{\partial x^2} - \nu\frac{\partial v(t,x)}{\partial x} - \mu_0 v(t,x) \\
&\quad + e^{\lambda\tau}\int_0^L \Gamma(\tau,x,y)f(R^*(t-\tau,y))v(t-\tau,y)dy + \lambda v(t,x),
\end{aligned}$$

for all $(t,x) \in (0,\infty) \times (0,L)$. Thus, $v(t,x)$ is a solution of the following ω-periodic equation:

$$\begin{cases}
\dfrac{\partial v(t,x)}{\partial t} = \delta\dfrac{\partial^2 v}{\partial x^2} - \nu\dfrac{\partial v}{\partial x} - \mu_0 v + e^{\lambda\tau}\displaystyle\int_0^L \Gamma(\tau,x,y) & \\
\qquad\qquad \times f(R^*(t-\tau,y))v(t-\tau,y)dy + \lambda v, & t > 0,\ \ x \in (0,L), \\
\nu v(t,0) - \delta\dfrac{\partial v}{\partial x}(t,0) = \dfrac{\partial v}{\partial x}(t,L) = 0, & t > 0, \\
v(s,x) = e^{\lambda s}\bar{\phi}(s,x), & s \in [-\tau,0],\ \ x \in (0,L).
\end{cases} \tag{3.2}$$

For any $\theta \in [-\tau, 0]$, $x \in (0, L)$, we have

$$v(\omega+\theta, x) = e^{\lambda(\omega+\theta)} \cdot P(\bar{\phi})(\theta, x) = e^{\lambda(\omega+\theta)} \cdot r_0\bar{\phi}(\theta, x) = e^{\lambda\theta}S(\theta, x; \bar{\phi}) = v(\theta, x).$$

Therefore, $v_0(\theta, \cdot) = v_\omega(\theta, \cdot)$ for all $\theta \in [-\tau, 0]$, and hence, the existence and uniqueness of solutions of (3.2) imply that $v(t, x) = v(t + \omega, x), \forall t \geq -\tau, x \in (0, L)$, i.e. $v(t, x)$ is an ω-periodic solution of (3.2). Clearly, $e^{-\lambda t}v(t, x)$ is a solution of (3.1). □

Before proving the main result on the threshold dynamics of system (1.7), we need the following lemma.

Lemma 3.2. *Suppose* $(R(t, x; \phi), u(t, x; \phi))$ *is the solution of system* (1.7) *with initial data* $\phi = (R_0, u_0) \in C_\tau^+$.

(i) *If there exists some* $t_0 \geq 0$ *such that* $u(t_0, \cdot; \phi) \not\equiv 0$, *then* $u(t, x; \phi) > 0$ *for all* $t > t_0$ *and* $x \in [0, L]$;
(ii) *For any* $\phi \in C_\tau^+$, *we have* $R(t, \cdot; \phi) > 0, \forall t > 0$ *and*

$$\liminf_{t\to\infty} R(t, x; \phi) \geq \eta,$$

uniformly for $x \in [0, L]$, *where* η *is a positive constant.*

Proof. From Theorem 2.2 and the second equation of (1.7), it is easy to see that $u(t, x; \phi)$ satisfies

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} \geq \delta\dfrac{\partial^2 u}{\partial x^2} - \nu\dfrac{\partial u}{\partial x} - \mu_0 u, & t > 0, \ \ x \in (0, L), \\ \nu u(t, 0) - \delta\dfrac{\partial u}{\partial x}(t, 0) = \dfrac{\partial u}{\partial x}(t, L) = 0, & t > 0. \end{cases}$$

If $u(t_0, \cdot; \phi) \not\equiv 0$ for some $t_0 \geq 0$, it then follows from the parabolic maximum principle that $u(t, x; \phi) > 0$ for all $t > t_0, x \in [0, L]$. Thus, part (i) holds.

By Theorem 2.2, there is a constant $B > 0$ such that $u(t, x) \leq B$ for $t > 0$ and $x \in [0, L]$. In view of (1.4), there exists $M > 0$ such that $0 \leq f(R) \leq MR, \forall R \geq 0$. Set $d := qBM$. Let $v(t, x; \phi)$ be the solution of

$$\begin{cases} \dfrac{\partial v(t,x)}{\partial t} = \delta\dfrac{\partial^2 v}{\partial x^2} - \nu\dfrac{\partial v}{\partial x} - dv, & & t > 0, \ \ x \in (0, L), \\ \nu v(t, 0) - \delta\dfrac{\partial v}{\partial x}(t, 0) = \nu R^{(0)}(t), & \dfrac{\partial v}{\partial x}(t, L) = 0, & t > 0, \\ v(0, x) = \phi_1(0, x), & & x \in (0, L). \end{cases} \tag{3.3}$$

By similar arguments to [26, Proposition 2.1], we can show that system (3.3) admits a unique positive ω-periodic solution $v^*(t, x)$ which is globally attractive in $\mathbb{Y}^+$.

In view of the comparison principle, it follows that

$$R(t,x;\phi)\geq v(t,x;\phi_1),\quad t>0,\ x\in[0,L].$$

Furthermore, by the maximum principle, we have

$$\liminf_{t\to\infty} R(t,x;\phi)\geq \inf_{t\in[0,\omega],x\in[0,L]} v^*(t,x)\quad \text{uniformly for } x\in[0,L],$$

where $v^*(t,x)$ is the unique positive ω-periodic solution of (3.3). Thus, part (ii) is proved. □

Theorem 3.3. *Let $(R(t,x;\phi),u(t,x;\phi))$ be the solution of (1.7) with the initial data $\phi=(R_0,u_0)\in C_\tau^+$. Then the following statements hold*:

(i) *If $r_0<1$, then*

$$\lim_{t\to\infty}(R(t,x;\phi)-R^*(t,x))=0\quad\text{and}\quad \lim_{t\to\infty}u(t,x)=0$$

uniformly for $x\in[0,L]$, where $R^*(t,x)$ is defined in Lemma 2.1;

(ii) *If $r_0>1$, then system (1.7) admits at least one positive ω-periodic solution $(\tilde R(t,x),\tilde u(t,x))$, and there exists $\zeta>0$ such that for any $\phi\in C_\tau^+$ with $u_0(0,\cdot)\not\equiv 0$, we have*

$$\liminf_{t\to\infty}(R(t,x;\phi),u(t,x;\phi))\geq(\zeta,\zeta)$$

uniformly for all $x\in[0,L]$.

Proof. (i) In the case where $r_0<1$, we have $\lambda=-\frac{1}{\omega}\ln r_0>0$ by Lemma 3.1. Consider the following system with parameter $\epsilon>0$:

$$\begin{cases}\dfrac{\partial u_\epsilon}{\partial t}=\delta\dfrac{\partial^2 u_\epsilon}{\partial x^2}-\nu\dfrac{\partial u_\epsilon}{\partial x}-\mu_0 u_\epsilon+\displaystyle\int_0^L\Gamma(\tau,x,y) & \\ \qquad\qquad\times f(R^*(t-\tau,y)+\epsilon)u_\epsilon(t-\tau,y)dy, & t>0,\ \ x\in(0,L),\\ \nu u_\epsilon(t,0)-\delta\dfrac{\partial u_\epsilon}{\partial x}(t,0)=\dfrac{\partial u_\epsilon}{\partial x}(t,L)=0, & t>0,\\ u_\epsilon(s,x)=\varphi(s,x), & s\in[-\tau,0],\ \ x\in(0,L).\end{cases}\tag{3.4}$$

Define the Poincaré map of (3.4) $P_\epsilon:\mathcal{E}\to\mathcal{E}$ by $P_\epsilon(\varphi)=u_{\epsilon,\omega}(\varphi)$ for $\varphi\in\mathcal{E}$, where $u_{\epsilon,\omega}(\varphi)(s,x)=u_\epsilon(\omega+s,x;\varphi),\forall(s,x)\in[-\tau,0]\times[0,L]$, and $u_\epsilon(t,x;\varphi)$ is the solution of (3.4) with $u_\epsilon(s,x)=\varphi(s,x)$ for $(s,x)\in[-\tau,0]\times(0,L)$. Let $r_\epsilon=r(P_\epsilon)$ be the spectral radius of P_ϵ. Thus, we can conclude from $r_0<1$ that there exists a sufficient small positive number ϵ_1 such that $r_\epsilon<1$ for all $\epsilon\in[0,\epsilon_1)$. Fix $\epsilon\in(0,\epsilon_1)$, we have $\lambda_\epsilon=-\frac{1}{\omega}\ln r_\epsilon>0$. In view of Lemma 3.1, there exists an ω-periodic function $v_\epsilon^*(t,x)$ such that $u_\epsilon(t,x)=e^{-\lambda_\epsilon t}v_\epsilon^*(t,x)$ is a solution of (3.4). In particular, $v_\epsilon^*(t,x)>0$ for any $t\in\mathbb{R}$ and $x\in[0,L]$.

Note that $R(t,x)$ satisfies

$$\begin{cases} \dfrac{\partial R(t,x)}{\partial t} \leq \delta \dfrac{\partial^2 R(t,x)}{\partial x^2} - \nu \dfrac{\partial R(t,x)}{\partial x}, & t>0,\ \ x \in (0,L), \\ \nu R(t,0) - \delta \dfrac{\partial R}{\partial x}(t,0) = \nu R^{(0)}(t), \quad \dfrac{\partial R}{\partial x}(t,L) = 0, & t > 0. \end{cases} \tag{3.5}$$

By the parabolic comparison principle and Lemma 2.1, it follows that there exists an integer $k > 0$ such that $R(t,x;\phi) \leq R^*(t,x) + \epsilon$, $\forall t \geq k\omega$, $x \in [0,L]$, $\epsilon > 0$. Therefore, for $t \geq k\omega$, we have

$$\begin{cases} \dfrac{\partial u}{\partial t} \leq \delta \dfrac{\partial^2 u}{\partial x^2} - \nu \dfrac{\partial u}{\partial x} - \mu_0 u + \displaystyle\int_0^L \Gamma(\tau,x,y) \\ \qquad\qquad \times f(R^*(t-\tau,y)+\epsilon) u(t-\tau,y)dy, & t \geq k\omega,\ \ x \in (0,L), \\ \nu u(t,0) - \delta \dfrac{\partial u}{\partial x}(t,0) = \dfrac{\partial u}{\partial x}(t,L) = 0, & t \geq k\omega. \end{cases} \tag{3.6}$$

Given $\phi \in C_\tau^+$, since $u(t,x;\phi)$ is globally bounded, then there exists some $\alpha > 0$ such that $u(t,x;\phi) \leq \alpha e^{-\lambda_\epsilon t} v_\epsilon^*(t,x)$, $\forall\, t \in [k\omega, k\omega+\tau], x \in [0,L]$. Therefore, the comparison theorem for abstract functional differential equation [17, Proposition 3] can be applied to (3.6) and (3.4) to obtain $u(t,x;\phi) \leq \alpha e^{-\lambda_\epsilon t} v_\epsilon^*(t,x)$, $\forall\, t \geq k\omega + \tau$, which implies $\lim_{t\to\infty} u(t,x;\phi) = 0$ uniformly for $x \in [0,L]$ due to $\lambda_\varepsilon > 0$. Therefore, the equation for R in system (1.7) is asymptotic to the following equation:

$$\begin{cases} \dfrac{\partial R}{\partial t} = \delta \dfrac{\partial^2 R}{\partial x^2} - \nu \dfrac{\partial R}{\partial x}, & t>0,\ \ x \in (0,L), \\ \nu R(t,0) - \delta \dfrac{\partial R}{\partial x}(t,0) = \nu R^{(0)}(t), \quad \dfrac{\partial R}{\partial x}(t,L) = 0, & t > 0. \end{cases} \tag{3.7}$$

Lemma 2.1 implies that $R^*(t,x)$ is a global attractive solution of (3.7). Next, we show that $\lim_{t\to\infty} R(t,x;\phi) = R^*(t,x)$ uniformly for $x \in [0,L]$ by the theory of internally chain transitive sets (see, e.g. [8, 35, 38]).

Let $\mathbf{P} = \Phi_\omega : C_\tau^+ \to C_\tau^+$ be the Poincaré map of (1.7), and $J = \bar{\omega}(\phi)$ be the omega limit set of $\phi \in C_\tau^+$ for $\mathbf{P}$, i.e.

$$J = \left\{ (\phi_1^*, \phi_2^*) \in C_\tau^+ : \exists \{n_k\} \to \infty \text{ such that } \lim_{k\to\infty} \mathbf{P}^{n_k}(\phi_1,\phi_2) = (\phi_1^*, \phi_2^*) \right\}.$$

By [8, Lemma 2.1] (or see [38, Lemma 1.2.1′]), it follows that J is a compact, invariant and internally chain transitive set for $\mathbf{P}$. Since $\lim\limits_{t\to\infty} u(t,x;\phi) = 0$ uniformly for $x \in [0,L]$, then there exists a set $J_1 \subset \mathcal{E}^+$ such that $J = J_1 \times \{\hat{0}\}$. By Lemma 3.2, we have $\hat{0} \notin J_1$.

For any $\varphi \in \mathcal{E}^+$, let $R(t, x; \varphi(0, \cdot))$ be the solution of (3.7) with initial value $R(0, x) = \varphi(0, x)$. Define

$$R_t(\theta, x; \varphi) = \begin{cases} R(t+\theta, x; \varphi(0)), & t+\theta > 0, t > 0, \theta \in [-\tau, 0], \\ \varphi(t+\theta, x), & t+\theta \le 0, t > 0, \theta \in [-\tau, 0]. \end{cases}$$

Then R_t defines a solution semiflow of (3.7) on $\mathcal{E}^+$. Let $\tilde{P} = R_\omega(\varphi)$. It then follows from Lemma 2.1 that $\bar{\omega}(\varphi) = \{R_0^*\}$, where $\bar{\omega}(\varphi)$ denotes the omega limit set for $\tilde{P}$, and $R_0^* \in \mathcal{E}^+$ is defined by $R_0^*(\theta, \cdot) = R^*(\theta, \cdot)$ for $\theta \in [-\tau, 0]$. Since $\mathbf{P}(J) = J$ and $u(t, x; (R_0, \hat{0})) \equiv 0$, we have $\mathbf{P}(J) = \tilde{P}(J_1) \times \{\hat{0}\} = J_1 \times \{\hat{0}\}$, and hence, $\tilde{P}(J_1) = J_1$. Consequently, J_1 is an internally chain transitive set for $\tilde{P}$. According to Lemma 2.1 and above discussion, we know that $\{R_0^*\}$ is globally attractive in $\mathcal{E}^+$. It then follows from [8, Theorem 3.1] that $J_1 = \{R_0^*\}$. Thus, $J = \{(R_0^*, \hat{0})\}$. By the definition of J, we have

$$\lim_{t\to\infty} \|(R(t, \cdot; \phi), u(t, \cdot; \phi)) - (R^*(t, \cdot), 0)\| = 0.$$

(ii) In the case where $r_0 > 1$, we have $\lambda = -\frac{1}{\omega} \ln r_0 < 0$. Let

$$\mathbf{W}_0 = \{(R_0, u_0) \in C_\tau^+ : u_0 \not\equiv 0\} \quad \text{and}$$

$$\partial \mathbf{W}_0 = C_\tau^+ \backslash \mathbf{W}_0 = \{(R_0, u_0) \in C_\tau^+ : u_0 \equiv 0\}.$$

By Lemma 3.2, we have $u(t, x; \phi) > 0$ for any $\phi \in \mathbf{W}_0, t > 0$ and $x \in [0, L]$. It follows that $\Phi_\omega^k(\mathbf{W}_0) \subseteq \mathbf{W}_0$, $\forall k \in \mathbb{N}$. Let $M_\partial = \{\phi \in \partial\mathbf{W}_0 : \Phi_\omega^k(\phi) \in \partial\mathbf{W}_0, \forall k \in \mathbb{N}\}$ and $J(\phi)$ be the omega limit set of the orbit $\gamma^+(\phi) = \{\Phi_\omega^k(\phi) : \forall k \in \mathbb{N}\}$. Set $M = (R_0^*, \hat{0})$. For any given $\phi \in M_\partial$, we have $\Phi_\omega^k(\phi) \in \partial\mathbf{W}_0, \forall k \in \mathbb{N}$. Thus $u(t, \cdot; \phi) \equiv 0, \forall t \ge 0$. Therefore, it follows from Lemma 2.1 that $\lim_{t\to\infty} \|R(t, \cdot; \phi) - R^*(t, \cdot)\| = 0$. Thus, we have $J(\phi) = \{M\}$ for any $\phi \in M_\partial$.

Consider the following system:

$$\begin{cases} \dfrac{\partial u^\rho(t,x)}{\partial t} = \delta \dfrac{\partial^2 u^\rho(t,x)}{\partial x^2} - \nu \dfrac{\partial u^\rho(t,x)}{\partial x} - \mu_0 u^\rho(t,x) + \displaystyle\int_0^L \Gamma(\tau, x, y) \\ \qquad \times f(R^*(t-\tau, y) - \rho) u^\rho(t-\tau, y) dy, \quad t > 0, \ x \in (0, L), \\ \nu u^\rho - \delta \dfrac{\partial u^\rho}{\partial x}(t, 0) = \dfrac{\partial u^\rho}{\partial x}(t, L) = 0, \quad t > 0, \\ u^\rho(s, x) = \varphi(s, x), \quad s \in [-\tau, 0], \ x \in [0, L]. \end{cases} \tag{3.8}$$

Define the Poincaré map of (3.8) $P_\rho : \mathcal{E} \to \mathcal{E}$ by $P_\rho(\varphi) = u_\omega^\rho(\varphi)$, where $u_\omega^\rho(\varphi)(s, x) = u^\rho(\omega + s, x; \varphi)$ for $(s, x) \in [-\tau, 0] \times [0, L]$, and $u^\rho(t, x; \varphi)$ is the solution of (3.8) with $u^\rho(s, x) = \varphi(s, x)$ for all $s \in [-\tau, 0], x \in [0, L]$. Since $r_0 > 1$, there exists a sufficiently small positive number ρ_1 such that $r_\rho = r(P_\rho) > 1$ for all $\rho \in [0, \rho_1)$, where $r(P_\rho)$ is the spectral radius of P_ρ. Fix a $\bar{\rho} \in (0, \rho_1)$. By the continuous dependence of solutions on the initial value, there exists $\rho_0 \in (0, \rho_1)$ such that

$$|(R(t,x;\phi),u(t,x;\phi))-(R^*(t,x),0)|<\bar{\rho},\quad \forall\, t\in[0,\omega],\ x\in[0,L] \tag{3.9}$$

if $|\phi(s,x)-(R^*(s,x),0)|<\rho_0, \forall s\in[-\tau,0], x\in[0,L]$. We now prove the following claim.

Claim. *M is a uniform weak repeller for $\mathbf{W}_0$ in the sense that*

$$\limsup_{k\to\infty}\|\Phi_\omega^k(\phi)-M\|\geq\rho_0,\quad \forall\,\phi\in\mathbf{W}_0.$$

Suppose by contradiction, there exists $\phi_0\in\mathbf{W}_0$ such that

$$\limsup_{k\to\infty}\|\Phi_\omega^k(\phi_0)-M\|<\rho_0.$$

Then there exists $k_0\in\mathbb{N}$ such that $|R(k\omega+s,x;\phi_0)-R^*(k\omega+s,x)|<\rho_0$ and $|u(k\omega+s,x;\phi_0)|<\rho_0$ for all $k\geq k_0, s\in[-\tau,0]$ and $x\in[0,L]$. It follows from (3.9) that $R(t,x;\phi_0)>R^*(t,x)-\bar{\rho}$ and

$$0<u(t,x;\phi_0)<\bar{\rho}, \tag{3.10}$$

for any $t>k_0\omega$ and $x\in[0,L]$. Then $u(t,x;\phi_0)$ satisfies

$$\begin{cases}\dfrac{\partial u(t,x)}{\partial t}\geq\delta\dfrac{\partial^2u(t,x)}{\partial x^2}-\nu\dfrac{\partial u(t,x)}{\partial x}-\mu_0u(t,x)+\displaystyle\int_0^L\Gamma(\tau,x,y)\\ \qquad\qquad\times f(R^*(t-\tau,y)-\bar{\rho})u(t-\tau,y)dy,\quad t\geq(k_0+1)\omega,\ \ x\in(0,L),\\ \nu u(t,0)-\delta\dfrac{\partial u}{\partial x}(t,0)=\dfrac{\partial u}{\partial x}(t,L)=0,\quad t\geq(k_0+1)\omega,\end{cases} \tag{3.11}$$

Let $\bar{\varphi}\in\mathcal{E}$ be the positive eigenfunction of $P_{\bar{\rho}}$ associated with $r_{\bar{\rho}}$ and $\lambda_{\bar{\rho}}=-\frac{1}{\omega}\ln r_{\bar{\rho}}<0$. Since $u(t,x;\phi_0)>0$ for all $t>\tau$, $x\in[0,L]$, then there exists a $\xi>0$ such that

$$u((k_0+1)\omega+s,x;\phi_0)\geq\xi\bar{\varphi}(s,x),\quad \forall\, s\in[-\tau,0],\ x\in[0,L].$$

It follows from (3.11) and the comparison principle that

$$u(t,x;\phi_0)\geq\xi u^{\bar{\rho}}(t-(k_0+1)\omega,x;\bar{\varphi})\quad \forall\, t\geq(k_0+1)\omega,\, x\in[0,L].$$

Therefore, we have

$$\begin{aligned}u(k\omega,x;\phi)&=\xi u^{\bar{\rho}}((k-k_0-1)\omega,x;\bar{\varphi})\\ &=\xi(r_{\bar{\rho}})^{(k-k_0-1)}\bar{\varphi}(0,x)\to+\infty\quad\text{as } k\to+\infty,\end{aligned}$$

which is a contradiction to (3.10).

By the above claim, M is an isolated invariant set for Φ_ω in $\mathbf{W}_0$, and $W^S(M)\cap\mathbf{W}_0=\emptyset$, where $W^S(M)$ is the stable set of M. By the acyclicity theorem on uniform persistence for maps (see, e.g. [38, Theorem 1.3.1 and Remark 1.3.1]), we have that

$\Phi_\omega : C_\tau^+ \to C_\tau^+$ is uniformly persistent with respect to $(\mathbf{W}_0, \partial\mathbf{W}_0)$, i.e. there exists a $\tilde{\varsigma} > 0$ such that

$$\liminf_{k\to\infty} d(\Phi_\omega^k(\phi), \partial\mathbf{W}_0) \geq \tilde{\varsigma}, \quad \forall\, \phi \in \mathbf{W}_0.$$

It then follows from [38, Theorem 3.1.1] that the periodic semiflow $\Phi_t : C_\tau^+ \to C_\tau^+$ is also uniformly persistent with respect to $(\mathbf{W}_0, \partial\mathbf{W}_0)$. Since $\mathbf{P}^{n_0}$ is compact, where $n_0 = \min\{n \in \mathbb{N}, n\omega > 2\tau\}$, it follows from [16, Theorem 4.5] with $\rho(x) = d(x, \partial\mathbf{W}_0)$ that $\mathbf{P} : \mathbf{W}_0 \to \mathbf{W}_0$ has a global attractor $\mathcal{A}_0$ and system (1.7) has an ω-periodic solution $(\tilde{R}(t,\cdot), \tilde{u}(t,\cdot))$ with $(\tilde{R}_t(\cdot)(\cdot), \tilde{u}_t(\cdot)(\cdot)) \in \mathbf{W}_0$.

In order to prove the practice uniform persistence, we use the arguments similar to [15, Theorem 4.1] or [35, Theorem 4.3]. Define a continuous function $p : C_\tau^+ \to [0,\infty)$ by

$$p(\phi) := \min_{x\in[0,L]} \phi_2(0,x), \quad \forall\, \phi = (\phi_1, \phi_2) \in C_\tau^+.$$

Since $\mathcal{A}_0 = \mathbf{P}(\mathcal{A}_0) = \Phi_\omega(\mathcal{A}_0)$, we have that $\phi_2(0,\cdot) > 0$ for all $\phi \in \mathcal{A}_0$. Let $\mathcal{B}_0 := \bigcup_{t\in[0,\omega]} \Phi_t(\mathcal{A}_0)$. It then follows that $\mathcal{B}_0 \subset \mathbf{W}_0$ and $\lim_{t\to\infty} d(\Phi_t(\phi), \mathcal{B}_0) = 0$ for all $\phi \in \mathbf{W}_0$. Since $\mathcal{B}_0$ is a compact subset of $\mathbf{W}_0$, we have $\min_{\phi\in\mathcal{B}_0} p(\phi) > 0$. Thus, there exists a $\zeta^* > 0$ such that $\liminf_{t\to\infty} u(t,\cdot;\phi) \geq \zeta^*$. Furthermore, in view of Lemma 3.2, there exists a $0 < \zeta < \zeta^*$ such that

$$\liminf_{t\to\infty} (R(t,\cdot;\phi), u(t,\cdot;\phi)) \geq (\zeta, \zeta), \quad \forall\, \phi \in \mathbf{W}_0,$$

that is, the persistence statement in (ii) is proved. This completes the proof. $\square$

4. Discussion

In this paper, we study an unstirred chemostat model of a single species consuming single resource, where the input concentration of nutrient is time-dependent function reflecting the seasonal or daily changes, and time delay in the growth response accounts for time which laps between uptake of nutrient and the assimilation of nutrient into viable biomass. For system (1.7) modeling the dynamics of single population, we first investigate the existence and uniqueness of solution by appealing to the theory of semigroup, then establish a threshold type result on the global dynamics in terms of the principal eigenvalue of the Poincaré map. The results have enriched the theoretical study of chemostat model to some extent.

The chemostat plays an important role in bioprocessing, such as ecology, microbiology, chemical engineering, and so forth. The model that we propose and analyze here may be quite naturally exposed in many real-world phenomena, for instance, in chemostats simplest form, the system approximates conditions for plankton growth in lakes with limiting nutrient. The influence of seasonal change and time delay on plankton growth is obvious, so it is necessary to consider these factors. Note that mixed microbial cultures are very common, then it is natural to model and study the microbial competition for limiting nutrients. We will consider two-species competition chemostat model with time delay as our further work.

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Commom Solutions for a Class of Simultaneous Pell Equations

Ran Yinxia*
Longnan Teachers College, College of Mathematics and Information Science, Longnan, China.

* Corresponding author. Tel.: 18993930265; email: ranyinxia@163.com
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Abstract: In recent years, the common solution of pell equations is a hot field in indefinite equations. For example, the equations 1) mentioned in the paper. However, due to the diverse forms of such equations, many scholars have done more studies on the smaller values of k and m, and the main conclusions are focused on the estimation of solutions under some special forms of D_1 and the specific values of D. So there is a lot of room for studying these kinds of equations. In this paper, we studied the common solution of the system of indefinite equations 2) mentioned in this paper by using the elementary method and the recursive property of solution sequence. If *D* is the case in this paper, the common solution of the equations is given.

Key words: The system of indefinite equations, pell equation, integer solution, common solution, odd prime.

1. Introduction

The Diophantine equation is the oldest branch in number theory, whose content is extremely abundant, and it has close connections with the algebraic number theory, the algebraic geometry, the combinatorics and so on. In the recent 30 years, this field also has developed too much. In such fields as the information encoding theory, the algebraic number theory and the diophantine analysis theory, many types of the results of higher diophantine equation are used, which make it necessary for us to study some basic types of the solutions of higher diophantine equation. We are familiar to study some basic types of the simple diophantine equation and quadratic diophantine equation, while with the solution of higher diophantine equation, there is no general conclusion, so it needs further discussing.

The Diophantine equation not only developed actively itself, but also was apply to else fields of Discerete Mathematics. It plays an important role in people's study and research to solve the actual problems. So many researchers study the Diophantine equation extensively and highly in the domestic and abroad. Along with the development of the Diophantine equation, Algebraic Number Theory obtained the first formation and developments. Currently, Algebracic Number Theory has become a branch of mathematics with abundant contents, is also an important tool of studying of the Diophantine equation.

In recent years, the common solution of pell equations

$$\begin{cases} x^2 - D_1 y^2 = k \\ y^2 - Dz^2 = m \end{cases} \tag{1}$$

is a hot field in indefinite equations. The main conclusions are as follows:

1) When k=1 and m=1,

the research results of the system focus on the scope and estimation of the solution, and the main conclusions are shown in [1], [2].

2) When k=1 and m=4,

a) If D_1 =2, for the solution of the system, the main conclusion is shown in [3]-[10];

b) If D_1 =6, it is shown in [11]-[15];

c) If D_1 =10 , it is shown in the main conclusion [16].

d) If D_1 =12, it is shown in the main conclusion [17]-[19].

e) If D_1 =30, it is shown in the main conclusion [20].

3) When k=1 and m=25,

a) If D_1 =23, the situation of the system is discussed in [21].

However, the pell equations

$$\begin{cases} x^2 - k(k+1)y^2 = k \\ y^2 - Dz^2 = 4 \end{cases} (k \in \mathrm{N}^*),$$

is one of the kind of the equations (1). When k=2, it is shown in [11]-[15], when k=3, it is shown in the main conclusion [17]-[19]. In this paper, we deal with the case of k=4, namely

$$\begin{cases} x^2 - 20y^2 = 1 \\ y^2 - Dz^2 = 4 \end{cases} \tag{2}$$

And the following conclusions are obtained:

Theorem If $D = 2^t p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_4} p_4^{\alpha_3}$, where $\alpha_s = 0$ or $1, p_s \left(1 \le s \le 4\right)$ are distinct odd primes, t is a positive integer, and the solution of the indefinite system (1) is as follows:

a) D=2×7×23, the system (2) has non-trivial solutions (x, y, z)=(±2889, ±646,±36);

b) D=2^3×7×23, the system (2) has non-trivial solutions (x, y, z)=(±2889, ±646,±18);

c) D=2^5×7×23, the system (2) has non-trivial solutions (x, y, z)=(±2889, ±646,±9).

d) When $t \ne 1,3,5$, the system (2) only has trivial solutions (x, y, z) =(±9, ±2,0).

2. Preliminaries

Lemma 1 [18] If p is an odd prime number, then the diophantine equation x^4-py^2=1 has no other positive integer solution except p=5, x=3, y=4 and p=29, x=99, y=1820.

Lemma 2 [18] If a is a square number and a >1, the equation $ax^4 - by^2 = 1$ has only one positive integer solution.

Lemma 3 [18] If D is a non-square positive integer, then $x^4 - Dy^4 = 1$ has at most two positive integer solutions. And the sufficient and necessary condition for the equation to have two groups of solutions is that D=1785 or D=28560, or that $2x_0$ and $2y_0$ are squares, where (x_0, y_0) is the fundamental solution of the equation.

Lemma 4 If x_n, y_n is any integer solution of Pell equation x^2-104y^2=1, then x_n, y_n has the following properties:

(I) $y_n^{\,2}-4=y_{n-1}y_{n+1}$

(II) $y_{2n}=2x_ny_n, x_{2n}=2x_n^{\,2}-1;$

(III) $x_{n+1}=9x_n+40y_n, y_{n+1}=2x_n+9y_n,$

$x_{n+2}=18x_{n+1}-x_n, y_{n+2}=18y_{n+1}-y_n,$

$x_0=1, x_1=5,\ \ y_0=0, y_1=2;$

(IV) $(x_n,y_n)=1, (x_n,x_{n+1})=1, (y_n,y_{n+1})=2;$

(V) $x_{2n}\equiv\pm1(\bmod 9),\ \ y_{2n}\equiv 0(\bmod 9);$

$x_{2n+1}\equiv 0(\bmod 9), y_{2n+1}\equiv\pm2(\bmod 9);$

$y_{2n}\equiv 0(\bmod 4),\ y_{2n+1}\equiv 2(\bmod 4).$

Lemma 5 If (x_1, y_1) is the fundamental solution of Pell equation x^2-20y^2=1, and all integer solutions are (x_n, y_n),$n\in$Z. For any (x_n, y_n),it has the following properties:

a) x_n is square if and only if n=0;

b) $\frac{x_n}{5}$ is square if and only if $n=\pm1$;

c) $\frac{y_n}{2}$ is square if and only if n=0, 1.

3. Proof of Theorem

Proof: Since the fundamental solution of Pell equation x^2-20y^2=1 is $(x_1,y_1)=(9,2)$, all integer solutions of pell equation are $x_n+y_n\sqrt{20}=\left(9+2\sqrt{20}\right)^n, n\in \mathrm{Z}$. Thus:

If $(x,y,z)=(x_n,y_n,z)$ is the integer solution to (2), then $\forall n\in \mathrm{Z}$,

$$y_n^{\,2}-4=y_n^{\,2}-4\left(x_n^{\,2}-20y_n^{\,2}\right)=81y_n^{\,2}-4x_n^{\,2}=(9y_n+2x_n)(9y_n-2x_n)=y_{n+1}y_{n-1} \tag{3}$$

By (2) $Dz^2=y_n^{\,2}-4$

Then

$$Dz^2=y_{n+1}y_{n-1} \tag{4}$$

case1 Let n be odd, might as well $n=2m-1,(m\in \mathrm{Z})$, At this point, equation (4) becomes:

$$Dz^2=y_{n-1}y_{n+1}=y_{2m-2}y_{2m}=4x_{m-1}y_{m-1}x_my_m \tag{5}$$

case1. 1 Let m be odd, might as well $m=2r,\left(r\in \mathrm{N}^*\right)$, At this point, equation (5) becomes:

$$Dz^2=4x_{2r-1}y_{2r-1}x_{2r}y_{2r}=8x_{2r-1}y_{2r-1}x_{2r}x_ry_r \tag{6}$$

case 1.1.1 Let r be odd, might as well $r=2u-1,(u\in Z)$, At this point, equation (5) becomes:

$$Dz^2 = 8x_{4u-3}y_{4u-3}x_{4u-2}x_{2u-1}y_{2u-1} \tag{7}$$

From lemma 5, $\frac{x_{2u-1}}{9}, \frac{x_{4u-3}}{9}, \frac{y_{2u-1}}{2}, \frac{y_{4u-3}}{2}, x_{4u-2}$ are two relatively prime, and $\frac{y_{2u-1}}{2}, \frac{y_{4u-3}}{2}$ are odd, $x_{4u-2}, \frac{x_{2u-1}}{9}, \frac{x_{4u-3}}{9}$ are odd, namely $\frac{x_{2u-1}}{9}, \frac{x_{4u-3}}{9}, \frac{y_{2u-1}}{2}, \frac{y_{4u-3}}{2}, x_{4u-2}$ are two relatively odd prime.

From lemma 5, that, if and only if u=0, 1, $\frac{x_{2u-1}}{9}$ is a square, and if and only if u=1, $\frac{x_{4u-3}}{9}$ is a square; For any l∈Z, x_{4u-2}, $\frac{y_{4u-1}}{2}$ it's not a square number. If and only if u=1, $\frac{y_{2u-1}}{2}, \frac{y_{4u-3}}{2}$ all are a square number. So if u≠0,1, $\frac{x_{2u-1}}{9}, \frac{x_{4u-3}}{9}, \frac{y_{2u-1}}{2}, \frac{y_{4u-3}}{2}, x_{4u-2}$ does not equal 0,1, it's not a square number.

When u=0, equation (7) is

$$Dz^2 = 2^5 \cdot 9^2 \cdot x_2 \cdot \frac{x_3}{9} \cdot \frac{y_3}{2} \tag{8}$$

However, $x_2 = 161 = 7\times 23, \frac{x_3}{9} = \frac{2889}{9} = 3\times 107, \frac{y_3}{2} = \frac{646}{2} = 17\times 19$

Therefore, the right hand side of (8) contains six different odd prime Numbers, so formula (8) does not hold, and the system (2) has no solution.

When u = 1,

$$Dz^2 = 8x_1y_1x_2x_1y_1 = 2^3\times 9^2\times 2^2\times 161 = 2^5\times 3^4\times 7\times 23 = 2\times 7\times 23\times\left(2^2\times 3^2\right)^2 = 2^3\times 7\times 23\times\left(2\times 3^2\right)^2 = 2^5\times 7\times 23\times 3^4 .$$

So when D=2×7×23, the system (2) has a nontrivial solutions (x, y, z) = (±2889,±646,±36); D=2^3×7×23, (2) has a nontrivial solution (x, y, z) = (±2889,±646,±18), when D=2^3×7×23, (2) has a nontrivial solution (x, y, z) = (±2889,±646,±9).

case 1.1.2 If r is even, let $r = 2v, (v \in Z)$, then equation (5) can be written into

$$Dz^2 = 8x_{4v-1}y_{4v-1}x_{4v}x_{2v}y_{2v} = 16x_{4v-1}y_{4v-1}x_{4v}x_{2v}x_vy_v \tag{9}$$

From lemma 5, when v is even, $\frac{x_{4v-1}}{9}, \frac{y_{4v-1}}{2}, x_{4v}, x_{2v}, x_v, \frac{y_v}{18}$ are two relatively prime, when v is odd $\frac{x_{4v-1}}{9}, \frac{y_{4v-1}}{2}, x_{4v}, x_{2v}, \frac{x_v}{9}, \frac{y_v}{2}$ are two relatively prime. And when v is odd, $\frac{y_{4v-1}}{2}, \frac{x_{4v-1}}{9}, \frac{x_v}{9}, x_{4v}, x_{2v}, \frac{x_v}{9}$ all are odd; when v is even, $\frac{y_{4v-1}}{2}, \frac{x_{4v-1}}{9}, \frac{x_v}{2}, x_{4v}, x_{2v}$ all are odd;

From lemma 5, if and only if v=0, $\frac{x_{4v-1}}{5}, x_{4v}, x_{2v}, x_v \frac{x_{2u-1}}{51}$ are squares, and if and only if v=±1, $\frac{x_v}{5}$ is a square; For any v∈Z, x_{4u-2}, $\frac{y_{4v-1}}{2}$ is not square. If and only if v=0, 1, $\frac{y_v}{2}$ is a square. So if v ≠0 and v is even

$x_{4v}, x_{2v}, x_v, \frac{x_{4v-1}}{9}, \frac{y_{4v-1}}{2}$ are not squares. At this point, they have at least five different odd prime Numbers, so formula (9) is not true, so when u≠0,1, the system (2) has no solution.

When v≠±1 and v is odd, $x_{4v}, x_{2v}, \frac{x_v}{9}, \frac{x_{4v-1}}{9}, \frac{y_{4v-1}}{2}$ are not squares. At this point, they have at least five different odd prime Numbers, so formula (9) is not true, so when u≠0,1, the system (2) has no solution. So when v≠0, v≠±1 and v is even, the system (2) has no solution.

When v=0, (9) can be written into $Dz^2 = 16 \cdot x_0^{\ 3} \cdot y_0 \cdot x_{-1} \cdot y_{-1} = 0$, thus z=0, At this point, the system (2), only has ordinary solutions (x, y, z)= (±9,±2, 0).

When v=1, (9) can be written into

$$\begin{aligned} Dz^2 &= 16 \cdot x_3 \cdot y_3 \cdot x_4 \cdot x_2 \cdot x_1 \cdot y_1 = 16 \times 3^3 \times 107 \times 2 \times 17 \times 19 \times 47 \times 1103 \times 7 \times 23 \times 9 \times 2, \\ &= 2^6 \times 3^5 \times 7 \times 17 \times 19 \times 23 \times 47 \times 107 \times 1103 \end{aligned}$$

The right hand side of the above equation contains eight odd prime Numbers, so the above formula is impossible. Therefor when v=1, the system (2) has no common solution.

When v=-1,

$$\begin{aligned} Dz^2 &= 16 x_{-5} y_{-5} x_{-4} x_{-2} x_{-1} y_{-1} = 16 \cdot 9^2 \cdot 2^2 \cdot x_2 \cdot x_4 \cdot \frac{x_5}{9} \cdot \frac{y_5}{2} \\ &= 16 \times 9^2 \times 2^2 \times 161 \times 51841 \times \frac{930249}{9} \times \frac{208010}{2} \\ &= 2^6 \times 3^4 \times 7 \times 23 \times 47 \times 1103 \times 41 \times 2521 \times 5 \times 11 \times 31 \times 61 \\ &= 2^6 \times 3^4 \times 5 \times 7 \times 11 \times 23 \times 31 \times 41 \times 47 \times 61 \times 1103 \times 2521 \end{aligned}$$

Therefore, the right hand side of the above equation contains ten odd prime Numbers, so the above formula is impossible. Therefor when v=-1, the system (2) has no common solution.

case 1.2 If m is odd, modelled on the case 1.1, it can be proved that the equation (2) is only the common solution (x, y, z)= (±9,±2,0).

case2 If n is even, by lemma 4, $y_{n-1} \equiv y_{n+1} \equiv 1 \pmod 2$, the right-hand side of equation (4) is odd, while the left-hand side is even in the form of D, so the system (2) has no common solution.

4. Summary and Prospect

In this paper, we have gotten the solutions of the following equations

$$\begin{cases} x^2 - k(k+1)y^2 = k \\ y^2 - Dz^2 = 4 \end{cases} (k \in \mathrm{N}^*),$$

When k=4 and $D = 2^t p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_4} p_4^{\alpha_3}$, where $\alpha_s = 0$ or $1\left(1 \le s \le 4\right)$. And we can go on and talk about the solutions to this system when k > 4, and D is some other form. So there is a lot of room for studying these kinds of equations.

Due to the diverse forms of such equations, many scholars have done more studies on the smaller values of k and m, and the main conclusions are focused on the estimation of solutions under some special forms of D_1 and the specific values of D. We need more powerful ways of finding common solutions to more forms of equations.

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Ran Yinxia was born in April 1983. She comes from Gansu province, China. She graduated from Northwestern University with a master of science degree in 2009, and studied on number theory research. After graduation, she has been taught in Longnan Teachers College.

On the Generalized LEBESGUE-Ramanujan-Nagell Equation

Ran Yinxia*
Longnan Teachers College, Department of Mathematics, Gansu Longnan, China.

* Corresponding author. Tel.: 18993930265; email: ranyinxia@163.com
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Abstract: Let p is a prime, we studied the the generalized Lebesgue-Ramanujan-Nagell equation. By using the elementary method and algebraic number theory, we obtain one necessary condition which the equation has integer solutions and some sufficient conditions which the equation has no integer solution. 1). Let x be an odd number, one necessary condition which the equation has integer solutions is that $2^{n(p-1)}$-1/p contains some square factors. 2). Let x be an even number, when n=pk(k⩾1), all integer solutions for the equation are (x,y)=(0,4^k); when n=pk+(p-1)/2(k⩾0), all integer solutions are (±$2^{pk+(p-1)/2}$,2^{2k+1}); when n≡1,2,3,…,(p-3)/2, (p+1)/2,...,p-1(mod$p$), the equation has no integer solution.

Key words: Exponential Diophantine equation, integer solutions, integer ring, algebraic number theory.

1. Introduction

Let **N**, **Z** be the set of all positive integers and all integers respectively. In this paper, we deal with the solutions (x, y) of diophantine equation

$$Ax^2 + B = y^m, \quad m \equiv 1 \pmod 2, \quad m > 1, \quad x, y, m \in \mathrm{N} \tag{1}$$

where A ,B are positive integers and A is nonsquare. Some special cases of (1) have been settled. When A=1, B=1 lebsgue [1] has proved that (1) has no integer solution, when $A = 2, B = 1, n = 5$, Nagell [2] has proved that (1) has only integer solutions $(x, y) = (\pm 11, 3)$; When $A = 1, B = 4^n, m = 7$, and n=1, 2, 3, 4 (see [3]-[6]), it has been proved that (1) has no integer solution.

However, when $B = c^k$, it is more difficult to solve it. In particular, when $B = p^k$, It is a hot research field recently. And, at present these research results were achieved as follow:

1) When p=2, Cohn[1,2], Arif and Abu Muriefah[3], Le[4] have gotten all solutions of the equation $x^2 + 2^m = y^n, \gcd(x, y) = 1, n > 2$:

- When m is odd, the equation has only two solutions $(x, y, m, n) = (5, 3, 1, 3)$ and $(7, 3, 5, 4)$.
- When m is even, the equation has only one solution $(x, y, m, n) = (11, 5, 2, 3)$.

2) When p=3, Cohn[1,2], Arif and Abu Muriefah[5,6],luca[7],Tao[8] have gotten all solutions of the

equation $x^2+3^m=y^n,(x,y)=1,n>2$.

3) When *p*=5, Arif and Abu Muriefah [9], [10] and Tao [11] have gotten all solutions of the equation $x^2+5^m=y^n,(x,y)=1,n>2, and\, 2\mid m$. Unfortunately, it failed to give the solutions of $2\mid m$.

4) When *p*=7, Silksek and Cremona [12], Bugeaud, Mignotte and Silksek [13], Luca [14], Huilin-Zhu and Maohua-Le [15] have gotten all solutions of the equation $x^2+5^7=y^n$, $(x,y)=1,n>2$, and $2\mid m$. And, when $2\mid m$, they only got the solutions of $p=11,19,43,67,163$.

Here, we study the solution of $x^2+4^n=y^p$, where p is a prime, and give the following conclusions:

Theorem When $A=1,B=4^n,m=p$, the following conclusions will be established:

1) Let x be an odd number, one necessary condition which the equation (1) has integer solutions is that $2^{n(p-1)}-1/p$ contains some square factors.

2) Let x be an even number, if $n\equiv 0\pmod p$, that is $n=pk\,(k\geq 1)$, all integer solutions for the equation are $(x,y)=(0,4^k)$; if $n\equiv \frac{p-1}{2}\pmod p$, that is $n=pk+\frac{p-1}{2}\,(k\geq 0)$, all integer solutions are $\left(\pm 2^{pk+\frac{p-1}{2}},2^{2k+1}\right)$; if $n\equiv 1,2,3,\cdots,\frac{p-3}{2},\frac{p+1}{2},\cdots p-1\pmod p$, the equation has no integer solution.

2. Preliminaries

Lemma 1 [7] Let M is a unique factorization domain, k is a positive integer, $k\geq 2$, and $\alpha,\beta\in M$, $(\alpha,\beta)=1$, and if $\alpha\beta=\gamma^k,\gamma\in M$, then $\alpha=\varepsilon_1\mu^k,\beta=\varepsilon_2\nu^k,\mu,\nu\in M$, and $\varepsilon_1\varepsilon_2=\varepsilon^k$, where $\varepsilon_1,\varepsilon_2,\varepsilon$ are units in *M*.

Lemma 2 For the diophantine equation $x^2+1=2^k y^p$, there are following conclusions:

1) If $k=0$, then the equation only has integer solution $(x,y)=(0,1)$;

2) If $k=1$, then the equation only has integer solutions $(x,y)=(\pm 1,1)$;

3) If $k=2,3,\cdots,p-1$, then all equations have no integer solutions.

proof: 1), 2) By lemma 1, it is easy to prove;

Obviously, x is an odd number, then $x^2\equiv 1\pmod 4$ and $x^2+1\equiv 2\pmod 4$, But if $k=2,3,\cdots,p-1$, then $x^2+1=2^k y^p\equiv 0\pmod 4$, This is a contradiction. So $x^2+1=2^k y^p$, $(k=2,3,\cdots,p-1)$ has no integer solutions.

Lemma 3 When $p\equiv 1\pmod 4$, if $k\equiv 0,1\pmod 4$, then $C_p^k(k\geq 0,k\in Z)$ is odd numbers, and if $k\equiv 2,3\pmod 4$, then $C_p^k(k\geq 0,k\in Z)$ is even numbers; when $p\equiv 3\pmod 4$, if $k\equiv 1,3\pmod 8$, then $C_p^k(k\geq 0,k\in Z)$ is odd numbers, and if $k\equiv 5,7\pmod 8$, then $C_p^k(k\geq 0,k\in Z)$ is even numbers.

Lemma 4 If *p* is a prime, and (*a*, *p*)=1, then $a^{p-1}\equiv 1\pmod p$.

3. Proof of Theorem

1) First, suppose $x\equiv 1(\bmod 2)$, in $Z[i]$, $x^2+4^n=y^p$ can be decomposed into as follows

$$\left(x+2^n i\right)\left(x-2^n i\right)=y^p, x, y\in Z.$$

Let $\delta=\left(x+2^n i, x-2^n i\right)$, because of $\delta\mid\left(2x, 2^{n+1}i\right)=2$, δ can only be $1, 1+i, 2$. But $x\equiv 1(\bmod 2)$, so $x+2^n\equiv 1(\bmod 2)$, then $\delta\neq 2$. If $\delta=1+i$, then $2=N(1+i)\mid N\left(x+2^n i\right)=x^2+2^{2n}$. However $x\equiv 1(\bmod 2)$, So the integer x does not exist. As a result, $\delta=1$. **Thus ,by lemma 1,** $x+2^n i=(a+bi)^p, x, a, b\in Z,$

If $p\equiv 1(\bmod 4)$, then

$$x=a^p-C_p^2a^{p-2}b^2+C_p^4a^{p-4}b^4-C_p^6a^{p-6}b^6+\cdots-C_p^{p-7}a^7b^{p-7}+C_p^{p-5}a^5b^{p-5}-C_p^{p-3}a^3b^{p-3}+C_p^{p-1}ab^{p-1};$$

$$2^n=b\left(C_p^1a^{p-1}-C_p^3a^{p-3}b^2+C_p^5a^{p-5}b^4-C_p^7a^{p-7}b^6+\cdots+C_p^{p-4}a^4b^{p-5}-C_p^{p-2}a^2b^{p-3}+b^{p-1}\right).$$

If $p\equiv 3(\bmod 4)$, then

$$x=a^p-C_p^2a^{p-2}b^2+C_p^4a^{p-4}b^4-C_p^6a^{p-6}b^6+\cdots+C_p^{p-7}a^7b^{p-7}-C_p^{p-5}a^5b^{p-5}+C_p^{p-3}a^3b^{p-3}-C_p^{p-1}ab^{p-1};$$

$$2^n=b\left(C_p^1a^{p-1}-C_p^3a^{p-3}b^2+C_p^5a^{p-5}b^4-C_p^7a^{p-7}b^6+\cdots-C_p^{p-4}a^4b^{p-5}+C_p^{p-2}a^2b^{p-3}-b^{p-1}\right).$$

So $b=\pm 1, \pm 2^t\,(1\leq t\leq n-1), \pm 2^n$.

If $b=\pm 1$, When $p\equiv 1(\bmod 4)$,

then $C_p^1a^{p-1}-C_p^3a^{p-3}+C_p^5a^{p-5}-C_p^7a^{p-7}+\cdots+C_p^{p-4}a^4-C_p^{p-2}a^2=\pm 2^n-1$, so a must be odd.

Let $p=4k+1$, by lemma3, $C_p^1, C_p^5, C_p^9, \cdots C_p^{p-8}, C_p^{p-4}$, these k integer numbers are odd, and $C_p^3, C_p^7, C_p^{11}, \cdots C_p^{p-6}, C_p^{p-2}$, these k integer numbers are even. Thus, if k is even, the equation $C_p^1a^{p-1}-C_p^3a^{p-3}+C_p^5a^{p-5}-C_p^7a^{p-7}+\cdots+C_p^{p-4}a^4-C_p^{p-2}a^2=\pm 2^n-1$ doesn't set up; and if k is odd, $x=a^p-C_p^2a^{p-2}b^2+C_p^4a^{p-4}b^4-C_p^6a^{p-6}b^6+\cdots-C_p^{p-7}a^7b^{p-7}+C_p^{p-5}a^5b^{p-5}-C_p^{p-3}a^3b^{p-3}+C_p^{p-1}ab^{p-1}$ is even, this contradict with $x\equiv 1(\bmod 2)$, in fact, $C_p^0, C_p^4, C_p^8, \cdots C_p^{p-5}, C_p^{p-1}$, these k+1 integer numbers are odd, $C_p^2, C_p^6, C_p^{10}, \cdots C_p^{p-7}, C_p^{p-3}$, these k integer numbers are even, so x is even.

When $p\equiv 3(\bmod 4)$, then $C_p^1a^{p-1}-C_p^3a^{p-3}+C_p^5a^{p-5}-C_p^7a^{p-7}+\cdots-C_p^{p-4}a^4+C_p^{p-2}a^2=\pm 2^n-1$, so a must be odd.

Let $p=8k+3$, by lemma3, $C_p^1, C_p^3, C_p^9, C_p^{11}, C_p^{17}, C_p^{19}, \cdots, C_p^{p-12}, C_p^{p-10}, C_p^{p-4}, C_p^{p-2}$ are odd integer numbers, and $C_p^5, C_p^7, C_p^{13}, C_p^{15}\cdots C_p^{p-8}, C_p^{p-6}$ are even integer numbers. Thus,

$C_p^1a^{p-1}-C_p^3a^{p-3}+C_p^5a^{p-5}-C_p^7a^{p-7}+\cdots-C_p^{p-4}a^4+C_p^{p-2}a^2$ is even, however, $\pm 2^n-1$ is odd.

Let $p=8k+7$, by lemma3, $C_p^1, C_p^3, C_p^9, C_p^{11}, C_p^{17}, C_p^{19}, \cdots, C_p^{p-12}, C_p^{p-10}, C_p^{p-4}, C_p^{p-2}$ are odd integer numbers, and $C_p^5, C_p^7, C_p^{13}, C_p^{15} \cdots, C_p^{p-9}, C_p^{p-8}, C_p^{p-2}$ are even integer numbers. Thus, $C_p^1 a^{p-1} - C_p^3 a^{p-3} + C_p^5 a^{p-5} - C_p^7 a^{p-7} + \cdots - C_p^{p-4} a^4 + C_p^{p-2} a^2$ is even, however, $\pm 2^n - 1$ is odd.

If $b = \pm 2^t \left(1 \le t \le n-1\right)$, then $C_p^1 a^{p-1} - C_p^3 a^{p-3} b^2 + C_p^5 a^{p-5} b^4 - C_p^7 a^{p-7} b^6 + \cdots \pm C_p^{p-4} a^4 b^{p-5} \mp C_p^{p-2} a^2 b^{p-3} \pm b^{p-1} = \pm 2^{n-t}$, so a is even. Thus $x = a^p - C_p^2 a^{p-2} b^2 + C_p^4 a^{p-4} b^4 - C_p^6 a^{p-6} b^6 + \cdots \mp C_p^{p-7} a^7 b^{p-7} \mp C_p^{p-5} a^5 b^{p-5} \mp C_p^{p-3} a^3 b^{p-3} \pm C_p^{p-1} ab^{p-1}$ is even, this contradict with $x \equiv 1 \left(\bmod 2\right)$;

If $b = -2^n$, When $p \equiv 1 \left(\bmod 4\right)$,

$C_p^1 a^{p-1} - C_p^3 a^{p-3} b^2 + C_p^5 a^{p-5} b^4 - C_p^7 a^{p-7} b^6 + \cdots + C_p^{p-4} a^4 b^{p-5} - C_p^{p-2} a^2 b^{p-3} + b^{p-1} = -1$ that is $C_p^1 a^{p-1} - C_p^3 a^{p-3} b^2 + C_p^5 a^{p-5} b^4 - C_p^7 a^{p-7} b^6 + \cdots + C_p^{p-4} a^4 b^{p-5} - C_p^{p-2} a^2 b^{p-3} = -1 - 2^{(p-1)n}$, so $2^{(p-1)n} \equiv -1 \left(\bmod p\right)$, but ,indeed, by lemma 4, $2^{(p-1)n} \equiv 1 \left(\bmod p\right)$;

When $p \equiv 3 \left(\bmod 4\right)$,

$$C_p^1 a^{p-1} - C_p^3 a^{p-3} b^2 + C_p^5 a^{p-5} b^4 - C_p^7 a^{p-7} b^6 + \cdots - C_p^{p-4} a^4 b^{p-5} + C_p^{p-2} a^2 b^{p-3} - b^{p-1} = -1,$$

that is $C_p^1 a^{p-1} - C_p^3 a^{p-3} b^2 + C_p^5 a^{p-5} b^4 - C_p^7 a^{p-7} b^6 + \cdots - C_p^{p-4} a^4 b^{p-5} + C_p^{p-2} a^2 b^{p-3} = 2^{(p-1)n} - 1$,

so $a^2 \left(a^{p-3} - \frac{C_p^3}{p} a^{p-5} b^2 + \frac{C_p^5}{p} a^{p-7} b^4 - \frac{C_p^7}{p} a^{p-9} b^6 + \cdots - \frac{C_p^{p-4}}{p} a^2 b^{p-5} + \frac{C_p^{p-2}}{p} ab^{p-3} \right) = 2^{(p-1)n-1} \big/ p$,

thus only when $2^{(p-1)n-1} \big/ p$ contains some square factors, the equation may have integer solutions.

If $b = 2^n$, When $p \equiv 1 \left(\bmod 4\right)$,

$$C_p^1 a^{p-1} - C_p^3 a^{p-3} b^2 + C_p^5 a^{p-5} b^4 - C_p^7 a^{p-7} b^6 + \cdots + C_p^{p-4} a^4 b^{p-5} - C_p^{p-2} a^2 b^{p-3} + b^{p-1} = 1,$$

that is $C_p^1 a^{p-1} - C_p^3 a^{p-3} b^2 + C_p^5 a^{p-5} b^4 - C_p^7 a^{p-7} b^6 + \cdots + C_p^{p-4} a^4 b^{p-5} - C_p^{p-2} a^2 b^{p-3} = 1 - 2^{(p-1)n}$,

so $-a^2 \left(a^{p-3} - \frac{C_p^3}{p} a^{p-5} b^2 + \frac{C_p^5}{p} a^{p-7} b^4 - \frac{C_p^7}{p} a^{p-9} b^6 + \cdots + \frac{C_p^{p-4}}{p} a^2 b^{p-5} - \frac{C_p^{p-2}}{p} ab^{p-3} \right) = 2^{(p-1)n-1} \big/ p$,

thus only when $2^{(p-1)n-1} \big/ p$ contains some square factors, the equation may have integer solutions.

When $p \equiv 3 \left(\bmod 4\right)$,

$$C_p^1 a^{p-1} - C_p^3 a^{p-3} b^2 + C_p^5 a^{p-5} b^4 - C_p^7 a^{p-7} b^6 + \cdots - C_p^{p-4} a^4 b^{p-5} + C_p^{p-2} a^2 b^{p-3} - b^{p-1} = 1,$$

that is $C_p^1 a^{p-1} - C_p^3 a^{p-3} b^2 + C_p^5 a^{p-5} b^4 - C_p^7 a^{p-7} b^6 + \cdots - C_p^{p-4} a^4 b^{p-5} + C_p^{p-2} a^2 b^{p-3} = 2^{(p-1)n} + 1$,

so $2^{(p-1)n} \equiv -1 \pmod p$, but, indeed, by lemma 4, $2^{(p-1)n} \equiv 1 \pmod p$;

So, when $x \equiv 1 \pmod 2$, one necessary condition which the equation has integer solutions is that $2^{(p-1)n-1} / p$ contains some square factors.

2) Second, suppose $x \equiv 0 \pmod 2$, thus $y \equiv 0 \pmod 2$. Now make $x = 2x_1, y = 2y_1$, then the equation can be turned into $x_1^2 + 4^{n-1} = 2^{p-2} y_1^p$, obviously $x_1 \equiv 0 \pmod 2$, then make $x_1 = 2x_2$, it can be $x_2^2 + 4^{n-2} = 2^{p-4} y_1^p$, also make $x_2 = 2x_3$ again, it can be $x_3^2 + 4^{n-3} = 2^{p-6} y_1^p$, ..., make $x_{\frac{p-3}{2}} = 2x_{\frac{p-1}{2}}$ again, it can be $x_{\frac{p-1}{2}}^2 + 4^{n-\frac{p-1}{2}} = 2y_1^p$, now make $x_{\frac{p-1}{2}} = 2x_{\frac{p+1}{2}}, y_1 = 2y_2$ it can be $x_{\frac{p+1}{2}}^2 + 4^{n-\frac{p+1}{2}} = 2^{p-1} y_2^p$, then make $x_{\frac{p+1}{2}} = 2x_{\frac{p+3}{2}}$ again, it can be $x_{\frac{p+3}{2}}^2 + 4^{n-\frac{p+3}{2}} = 2^{p-3} y_2^p$, ..., make $x_{p-1} = 2x_p$ again, it can be $x_p^2 + 4^{n-p} = y_2^p$, where $x_1, x_2, \cdots, x_p, y_1, y_2 \in Z$.

According to such substituted method, it can be concluded:

When $n \equiv 1 \pmod p$, the original equation is equivalent to solving $x^2 + 4 = y^p$, and according to the above-mentioned regularity, it is finally equivalent to solving $x^2 + 1 = 2^{p-2} y^p$; When $n \equiv 2 \pmod p$, it is equivalent to solving $x^2 + 4^2 = y^p$, and according to the same regularity, it is finally equivalent to solving $x^2 + 1 = 2^{p-4} y^p$; When $n \equiv 3 \pmod p$, it is equivalent to solving $x^2 + 4^3 = y^p$, and according to the same regularity, it is finally equivalent to solving $x^2 + 1 = 2^{p-6} y^p$; ..., When $n \equiv \frac{p-1}{2} \pmod p$, it is equivalent to solving $x^2 + 4^{\frac{p-1}{2}} = y^p$, and according to the same regularity, it is finally equivalent to solving $x^2 + 1 = 2y^p$; When $n \equiv \frac{p+1}{2} \pmod p$, it is equivalent to solving $x^2 + 4^{\frac{p+1}{2}} = y^p$, and according to the same regularity, it is equivalent to solving $x^2 + 1 = 2^{p-1} y^p$;..., When $n \equiv p-1 \pmod p$, it is equivalent to solving $x^2 + 4^{p-1} = y^p$, and according to the same regularity, it is finally equivalent to solving $x^2 + 1 = 2^2 y^p$; When $n \equiv 0 \pmod p$, the original equation is equivalent to solving $x^2 + 4^p = y^p$, and according to the same regularity, it is finally equivalent to solving $x^2 + 1 = y^p$. Therefore, by lemma2, when $n \equiv 1, 2, 3, \cdots, \frac{p-3}{2}, \frac{p-1}{2}, \cdots, p-1 \pmod p$, the equation has no integer solutions;

When $n \equiv 0, \frac{p-1}{2} \pmod p$, the equation has integer solutions, and when $n \equiv 0 \pmod p$ that is $n = pk\,(k \geq 1)$, solutions of the equation will must be $(x, y) = (0, 4^k)$; if $n \equiv \frac{p-1}{2} \pmod p$, that is

$n = pk + \frac{p-1}{2}(k \geq 0)$, all integer solutions are $\left(\pm 2^{pk+\frac{p-1}{2}}, 2^{2k+1}\right)$.

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Ran Yinxia was born in April 1983. She comes from Gansu province, China. She graduated from Northwestern University with a master of science degree in 2009, and studied on number theory research. After graduation, she has been taught in Longnan Teachers College.

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n-tilting Torsion Classes and n-cotilting Torsion-free Classes

HE Dong-lin, LI Yu-yan

(*Department of Mathematics, Longnan Teachers College, Longnan,* 742500, *China*)

Abstract: In this paper, we consider some generalizations of tilting torsion classes and cotilting torsion-free classes, give the definition and characterizations of n-tilting torsion classes and n-cotilting torsion-free classes, and study n-tilting preenvelopes and n-cotilting precovers.

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§1. Introduction

Tilting theory plays an important role in the representation of Artin algebra. The classical tilting modules were first considered in the early eighties by Brenner-Bulter [1] , Bongartz [2] and Happle and Ringel [3] etc. Beginning with Miyashita [4], the defining conditions for a classical tilting module were extented to arbitary rings or Abel categories by many authors, Wakamatsu [5], Colby and Fuller [6], Colpi and Trifaj [7], and recently, Angeleri Hgel and coelho [8], Bazzoni [9], Wei [10], Colpi and Fuller [11], and Di et al [12]. Among them, Miyashita [4] considered finitely generated tilting modules of finite projective dimension, while generalizations of tilting modules of projective dimension one over arbitary rings. In [7] an (not necessarily finitely generated) module T is said to be tilting (simplely, 1-tilting module) if $\mathrm{GenT} = \mathrm{T}^{\perp_1}$, where GenT is the class of modules which are epimorphic images of direct sums of copies of T and $T^{\perp_1}$ is the class of modules M such that $\mathrm{Ext}^1_{\mathrm{R}}(\mathrm{T},\mathrm{M}) = 0$. And it is proved that $t \subseteq R$-Mod is a tilting class if and only if $t = \mathrm{GenP}$ for a faithful, finendo, and t-projective module P. Angeleri

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Biographies: HE Dong-lin(1983-), female, native of Baiyin, Gangsu, M.S.D., Longnan Teachers College, engages in homological algebra.

and Trlifaj[13] discussed tilting preenvelopes and cotilting precovers. Meanwhile, tilting torsion classes (resp, cotilting torsion-free classes) were characterized as those pretorsion classes (resp, pretorsion-free classes) which provided special preenvelopes (resp, special precovers) for all modules. Bazzoin [9] considered (not necessarily finitely generated) tilting modules of projective dimension $\leq n$ (simply n-tilting modules), and proved that T is n-tilting module if and only if $\mathrm{Pres^nT} = \mathrm{T}^{\perp_{i\geq 1}}$. Dually, U is n-cotilting module if and only if $\mathrm{Copres^nT} = {}^{\perp_{i\geq 1}}\mathrm{T}$. It is nutural to consider n-tilting torsion classes and n-cotilting torsion-free classes, and to investigate n-tilting preenvelopes and n-cotilting precovers which are generalizations of tilting preenvelopes and cotilting precovers in [7].

The contents of this paper are summarized as follows. In section 2, we collect some known notions and results. In section 3, we introduce n-tilting torsion classes and discuss n-tilting preenvelopes. Furthermore, we give some characterizations of n-tilting torsion classes, and prove that: if t is n-tilting torsion classes, then t is envelope class. Section 4 is devoted to n-cotilting torsion-free classes and n-cotilting precovers.

§2. Preliminaries

Throughout this paper, R will be an associative ring with nonzero identity and all modules are unitary. Let R-Mod be the category of left R-modules and $T \in R$-Mod. We denote by $T^{\perp_{1\leq i\leq n}} := \{M \in R\text{-Mod} \mid \mathrm{Ext^i_R(T,M)} = 0 \text{ for all } 1 \leq i \leq n\}$, $T^{\perp_{i\geq 1}} := \{M \in R\text{-Mod} \mid \mathrm{Ext^i_R(T,M)} = 0 \text{ for all } i \geq 1\}$, and $T^{\perp_1} := \{M \in R\text{-Mod} \mid \mathrm{Ext^1_R(T,M)} = 0\}$. Dually, ${}^{\perp_{1\leq i\leq n}}T$, ${}^{\perp_{i\geq 1}}T$, and ${}^{\perp_1}T$ are defined similarly. Denote by AddT the class of modules isomorphic to direct summands of direct sums of copies of T and by $\mathrm{Pres^nT} := \{\mathrm{M} \in \mathrm{R\text{-}Mod} \mid$there exists an exact sequence $T_n \to \cdots \to T_2 \to T_1 \to M \to 0$ with $T_i \in \mathrm{AddT}$ for all $1 \leq i \leq n\}$. It is clear that $\mathrm{Pres^{n+1}T} \subseteq \mathrm{Pres^nT}$ and $\mathrm{Pres^1T} = \mathrm{GenT}$, where GenT denotes the class of all left R-modules generated by T. Dually, denote by ProdT the class of modules isomorphic to direct summands of direct products of copies of T, and by $\mathrm{Copres^nT} := \{\mathrm{M} \in \mathrm{R\text{-}Mod} \mid$there exists an exact sequence $0 \to M \to C_1 \to C_2 \to \cdots \to C_n$ with $C_i \in \mathrm{ProdT}$ for all $1 \leq i \leq n\}$. It is clear that $\mathrm{Copres^{n+1}T} \subseteq \mathrm{Copres^nT}$ and $\mathrm{Copres^1T} = \mathrm{CogenT}$, where CogenT denotes the class of all left R-modules cogenerated by T. T is a tilting module[7] provided that $\mathrm{GenT} = \mathrm{T}^{\perp_1}$. T is a cotilting module[14] provided that $\mathrm{CogenT} = {}^{\perp_1}T$. Let $x \subseteq R$-Mod, then x is a pretorsion class provided that x is closed under direct sums and factors. Moreover, x is a tilting torsion class provided that $x = \mathrm{GenT}$ for a tilting module T. Dually, x is pretorsion-free class provided that x is closed under direct products and submodules. x is a cotilting torsion-free class provided that $x = \mathrm{CogenU}$ for a cotilting module U.

Precovers and preenvelopes were first defined in [15] in the following manner: if y is a class of modules (closed under isomorphisms), a y-precover of R-module M is a morphism φ from Y ($Y \in y$) to M, such that $\mathrm{Hom_R(Y',\varphi)}$ is surjective for every $Y' \in y$. If in addition, any morphism $\alpha : Y \to Y$ verifying $\varphi \circ \alpha = \varphi$ is automorphism, then φ is said to be an y-cover. y is a precover (resp, cover) class provided that each R-module has a y-precover (resp, y-cover). y-preenvelope and y-envelope, preenvelope class and envelope class can be defined dually. An

R-module M is y-projective (resp, y-injective) provided that the functor $\mathrm{Hom_R(M,-)}$ (resp, $\mathrm{Hom_R(-,M)}$)is exact on short exact sequence of the form $0 \to U \to V \to W \to 0$, where $U, V, W \in y$. Denote by $y^{\perp} = \{M \in R\text{-Mod} \mid \mathrm{Ext}^1_{\mathrm{R}}(\mathrm{Y,M}) = 0$ for any $Y \in y$. ${}^{\perp}y$ is defined dually.

Definition 2.1[9] An R-module T is said to be n-tilting module if the following conditions are satisfied:
(1) $\mathrm{pd_R T \le n}$ (here $\mathrm{pd_R T}$ denotes the projective dimension of T).
(2) $\mathrm{Ext^i_R(T, T^{(\lambda)})} = 0$ for all $i \ge 1$ and all cardinal λ.
(3) There is a long exact sequence $0 \to R \to T_0 \to T_1 \to \cdots \to T_r \to 0$ with $T_i \in \mathrm{AddT}$ for every $0 \le i \le r$.

Dually, an R-module U is said to be n-cotilting module if it satisfy the following conditions:
(1) $\mathrm{id_R U \le n}$ (here $\mathrm{id_R U}$ denotes the injective dimension of U).
(2) $\mathrm{Ext^i_R(U^{\lambda}, U)} = 0$ for all $i \ge 1$ and all cardinal λ.
(3) There is a long exact sequence $0 \to U_r \to \cdots \to U_1 \to U_0 \to E \to 0$ with $U_i \in \mathrm{ProdT}$ for every $0 \le i \le r$.

Lemma 2.1[9] An R-module T is said to be n-tilting module if and only if $\mathrm{Pres^nT} = \mathrm{T}^{\perp_{i\ge1}}$. Dually, an R-module $U$is said to be n-cotilting module if and only if $\mathrm{CopresU} = {}^{\perp_{i\ge1}}U$.

Remark 2.1[9] Tilting modules in [7] are exactly 1-tilting modules, cotilting modules in [7] are exactly 1-cotilting modules.

Proposition 2.1 The following conditions are hold:
(1) If T is n-tilting module, then T is m-tilting module for any non-negative integer $m \ge n$.
(2) If T is n-tilting module, then $\mathrm{Pres^nT = Pres^{n+1}T = Pres^{n+2}T} = \cdots$.

Proof (1) Asumme that T is n-tilting module, then $\mathrm{Pres^nT} = \mathrm{T}^{\perp_{i\ge1}}$ by lemma 2.1. It is sufficient to prove that $\mathrm{Pres^nT = Pres^mT}$ for any non-negative integer $m \ge n$. If $m = n$, then it is clear that $\mathrm{Pres^nT = Pres^mT}$. If $m > n$, then $\mathrm{Pres^mT \subseteq Pres^nT}$ and $\mathrm{Pres^nT = Pres^{n+1}T}$ by [16, theorem 4.3]. For any $M \in \mathrm{Pres^nT = Pres^{n+1}T}$, there exists an exact sequence

$$T_{n+1} \to T_n \to \cdots \to T_2 \to T_1 \to M \to 0$$

with $T_i \in \mathrm{AddT}$ for all $1 \le i \le n+1$. Note that $K_1 = \mathrm{Kerf_1} \in \mathrm{Pres^nT}$, we can get $M \in \mathrm{Pres^{n+2}T}$. Repeat the process, and so on, it is easy to get $\mathrm{Pres^nT = Pres^mT}$. Therefore, if T is n-tilting module, then T is m-tilting module for any non-negative integer $m \ge n$.

(2)It is obvious following (1).

We can obtain the following proposition dually.

Proposition 2.2 For any R-module U and any non-negative integer n, the following conditions are hold:
(1) If U is n-cotilting module, then U is m-cotilting module for any non-negative integer $m \ge n$.
(2) If U is n-cotilting module, then $\mathrm{Copres^nU = Copres^{n+1}U = Copres^{n+2}U} = \ldots$.

§3. n-tilting Torsion Classes and n-tilting Preenvelopes

We start with the following definition.

Definition 3.1 Let $y \subseteq R$-Mod. y is said to be an n-tilting torsion class, if there exists an n- tilting module $T \in R$-Mod, such that $y = \mathrm{Pres^nT}$.

y is a 1-tilting torsion class, if and only if there exists a 1-tilting module T such that $y = \mathrm{Pres^1T} = \mathrm{GenT}$. It is clear that 1-tilting torsion classes are exactly tilting torsion classes in [7], 1-tilting torsion classes are n-tilting torsion classes. n-tilting torsion classes are generalizations of tilting torsion classes. According to [13,theorem2.1], tilting torsion classes are characterized as follows.

Lemma 3.1 Let R be a ring and $y \subseteq R$-Mod be a pretorsion class. Then the followings are equivalent:
(1) y is tilting torsion class;
(2) every module has a special y-preenvelope;
(3) there is a special y-preenvelope of R;
(4) there is a y-preenvelope of R, $b : R \to B$, such that b is injective and B is y-projective.

We now can state one of our main results by lemma 3.1 as follows.

Theorem 3.1 Let $y \subseteq R$-Mod be a pretorsion class. Then the followings are equivalent:
(1) y is n-tilting torsion class;
(2) for any R-module M, there is an exact sequence

$$0 \to M \to T_1 \xrightarrow{d_1} T_2 \xrightarrow{d_2} \cdots \to T_n \xrightarrow{d_n} I \to 0$$

with $T_i \in y$ and $\mathrm{Imd_i} \in {}^{\perp}y$ $(i = 1, 2, \ldots, n)$;
(3)there exists an exact sequence

$$0 \to R \to T_1 \xrightarrow{d_1} T_2 \xrightarrow{d_2} \cdots \to T_n \xrightarrow{d_n} I \to 0$$

with $T_i \in y$ and $\mathrm{Imd_i} \in {}^{\perp}y$ $(i = 1, 2, \ldots, n)$;
(4)there exists an exact sequence

$$0 \to R \to T_1 \xrightarrow{d_1} T_2 \xrightarrow{d_2} \cdots \to T_n \xrightarrow{d_n} I \to 0$$

with $T_i \in y$ and $\mathrm{Imd}_1 \in {}^{\perp}y$ and T_i is y-projective $(i = 1, 2, \ldots, n)$.

Proof (1) $\Rightarrow$ (2) Suppose y is n-tilting torsion class, then there exists an n-tilting module $T \in R$-Mod, such that $y = \mathrm{Pres^nT} = \mathrm{T}^{\perp_{i\geq 1}}$. For any cardinal k, we have $\mathrm{Ext^i_R(T, T^{(k)})} = 0$ by $T^k \in \mathrm{Pres^nT}$. According to [17, lemma6.8], for any module M, there is a y-torsion resolution of M of the form $0 \to M \to T_1 \to T^{(\lambda_1)} \to 0$, where $T_1 \in y$, λ_1 is a cardinal, $\mathrm{Ext^i_R(T^{(\lambda_1)}, N)} = 0$ for all $N \in y$. Repeat the process of M for $T^{(\lambda_1)}$, and so on, we can get an exact sequence

$$0 \to M \to T_1 \xrightarrow{d_1} T_2 \xrightarrow{d_2} \cdots \to T_n \xrightarrow{d_n} I \to 0$$

in which $T_i \in y$ and $\mathrm{Imd_i} \in {}^{\perp}y$ $(i = 1, 2, \ldots, n)$.

(2) ⇒ (3) It is obvious.

(3) ⇒ (4) Assume that there exists an exact sequence

$$0 \to R \to T_1 \xrightarrow{d_1} T_2 \xrightarrow{d_2} \cdots \to T_n \xrightarrow{d_n} I \to 0$$

with $T_i \in y$ and $\mathrm{Imd_i} \in {}^{\perp}y$ $(i = 1, 2, \ldots, n)$. It is only to prove T_i is y-projective $(i = 1, 2, \ldots, n)$. Consider the short exact sequence $0 \to \mathrm{Imd_{n-1}} \to \mathrm{T_n} \to \mathrm{I} \to 0$, since $I \cong \mathrm{Imd_n} \in {}^{\perp}y$ and $\mathrm{Imd_{n-1}} \in {}^{\perp}y$, We can get $T_n \in {}^{\perp}y$, which shows that $\mathrm{Hom_R(T_n, -)}$ is exact on any epimorphism with kernal in y. In particular, T_n is y-projective. Repeat the process to the short exact sequence $0 \to \mathrm{Imd_{n-2}} \to \mathrm{T_{n-1}} \to \mathrm{Imd_{n-1}} \to 0$, and so on, it is not difficult to obtain that T_i is y-projective $(i = 1, 2, \ldots, n)$.

(4) ⇒ (1) Assume that there exists an exact sequence

$$0 \to R \to T_1 \xrightarrow{d_1} T_2 \xrightarrow{d_2} \cdots \to T_n \xrightarrow{d_n} I \to 0$$

with $T_i \in y$ and $\mathrm{Imd_i} \in {}^{\perp}y$ $(i = 1, 2, \ldots, n)$. Consider the short exact sequence $0 \to R \to T_1 \to \mathrm{Imd_1} \to 0$, since $\mathrm{Imd_1} \in {}^{\perp}y$, so $R \to T_1$ is a y-preenvelope of R. Note that T_1 is y-projective and $R \to T_1$ is injective, we can obtain that y is 1-tilting torsion class by lemma 3.1. Therefore, y is n-tilting torsion class.

Theorem 3.2 Suppose y is an n-tilting torsion class in R-Mod. If there exists $x \subseteq R$-Mod which is closed under extensions, such that $\mathrm{AddP_x} \subseteq \mathrm{x} \subseteq {}^{\perp}y$ and $P_x^{\perp_1} = y$ for some $P_x \in y$. Then is an envelope class.

Proof Assume that there exists $x \subseteq R$-Mod which is closed under extensions, such that $P_x^{\perp_1} = y$ and $\mathrm{AddP_x} \subseteq \mathrm{x} \subseteq {}^{\perp}y$ for some $P_x \in y$. For any $M \in R$-Mod, since y is an n-tilting torsion class, we have $y = \mathrm{Pres^nT}$ for some n-tilting module T. Note that $\mathrm{Pres^nT}$ is closed under direct sums and $P_x^{\perp_1} = y$, so $P_x^{\lambda} \in y$ and $\mathrm{Ext^1_R(P_x, P_x^{(\lambda)})} = 0$ for all cardinals λ. Therefore, we obtain an exact sequence $\varepsilon : 0 \to M \to Y \to P_x^{(\alpha)} \to 0$ by [17,lemma 6.8], where $Y \in y$, α is a cardinal. Then ε is a generator for $\mathrm{Ext^1_R(Y, M)}$ in the sense of [18, proposition2.2.1]. According to our assumption and [18, theorem2.2.6], we know that M has an $x^{\perp}$-envelope. Since $y = ({}^{\perp}y)^{\perp} \subseteq x^{\perp} \subseteq (\mathrm{AddP_x})^{\perp} \subseteq \mathrm{P_x^{\perp_1}} = \mathrm{y}$, Then the conclusion is proved.

§4. n-cotilting Torsion-free Classes and n-cotilting Precovers

We start with the following definition.

Definition 4.1 Let $w \subseteq R$-Mod. w is said to be an n-cotilting torsion-free class, if there exists an n- cotilting module $U \in R$-Mod, such that $w = \mathrm{Copres^nU}$.

w is a 1-cotilting torsion-free class, if and only if there exists a 1-cotilting module U such that $w = \mathrm{Copres^1U} = \mathrm{CogenU}$. It is clear that 1-cotilting torsion-free classes are exactly cotilting torsion-free classes in [7], 1-cotilting torsion-free classes are n-cotilting torsion-free classes. n-cotilting torsion-free classes are generalizations of cotilting torsion-free classes. According to [13,theorem2.5], cotilting torsion-free classes are characterized as follows.

Lemma 4.1 Let R be a ring and $w \subseteq R$-Mod be a pretorsion-free class. Then the followings are equivalent:
(1) w is cotilting torsion-free class;
(2) every module has a special w-precover;
(3) there is a special w-precover of an injective cogenerator of R-Mod;
(4) there is a w-precover, $\pi : P \to E$, of an injective cogenerator E of R-Mod such that π is surjective and P is faithful and w-injective.

We now can state one of our main results by lemma 4.1 as follows.

Theorem 4.1 Let $w \subseteq R$-Mod be a pretorsion-free class. Then the followings are equivalent:
(1) w is n-cotilting torsion-free class;
(2) for any R-module M, there is an exact sequence

$$0 \to K \to W_n \xrightarrow{f_n} \cdots \to W_2 \xrightarrow{f_2} W_1 \xrightarrow{f_1} M \to 0$$

with $W_i \in w$ and $\mathrm{Kerf_i} \in w^{\perp}$ $(i = 1, 2, \ldots, n)$;
(3) there exists an exact sequence

$$0 \to K \to W_n \xrightarrow{f_n} \cdots \to W_2 \xrightarrow{f_2} W_1 \xrightarrow{f_1} E \to 0$$

with $W_i \in w$ and $\mathrm{Kerf_i} \in w^{\perp}$ $(i = 1, 2, \ldots, n)$, where E is an injective cogenerator of R-Mod;
(4) there exists an exact sequence

$$0 \to K \to W_n \xrightarrow{f_n} \cdots \to W_2 \xrightarrow{f_2} W_1 \xrightarrow{f_1} E \to 0$$

with $W_i \in w$ and $\mathrm{Kerf_1} \in w^{\perp}$, W_1 is faithful and W_i is w-injective $(i = 1, 2, \ldots, n)$, where E is an injective cogenerator of R-Mod;

Proof (1) $\Rightarrow$ (2) Suppose w is n-cotilting torsion-free class, then there exists an n-cotilting module $U \in R$-Mod, such that $w = \mathrm{Copres^n U} = {}^{\perp_{i\geq 1}}U$. By [19, lemma 2.14], for any module M, there is a w-torsion-free resolution of M of the form $0 \to U^{\lambda_1} \to W_1 \to M \to 0$, where $W_1 \in w$, λ_1 is a cardinal, $\mathrm{Ext^1_R(N, U^{\lambda_1})} = 0$ for all $N \in w$.

Repeat the process of M for U^{λ_1}, and so on, we can get an exact sequence

$$0 \to K \to W_n \xrightarrow{f_n} \cdots \to W_2 \xrightarrow{f_2} W_1 \xrightarrow{f_1} M \to 0$$

with $W_i \in w$ and $\mathrm{Kerf_i} = \mathrm{U^{\lambda_i}} \in w^{\perp}$ $(i = 1, 2, \ldots, n)$.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (4) Assume that there exists an exact sequence

$$0 \to K \to W_n \xrightarrow{f_n} \cdots \to W_2 \xrightarrow{f_2} W_1 \xrightarrow{f_1} E \to 0$$

with $W_i \in w$ and $\mathrm{Kerf_i} \in w^{\perp}$ $(i = 1, 2, \ldots, n)$, where E is an injective cogenerator of R-Mod. Consider the short exact sequence $0 \to K \to W_n \to \mathrm{Kerf_{n-1}} \to 0$, since $K \cong \mathrm{Kerf_n} \in w^{\perp}$ and $\mathrm{Kerf_{n-1}} \in w^{\perp}$, We can get $W_n \in w^{\perp}$, which shows that $\mathrm{Hom_R(-, W_n)}$ is exact on any monomorphism with cokernal in w. In particular, W_n is w-injective. Repeat the process to the short exact sequence $0 \to \mathrm{Kerf_{n-1}} \to \mathrm{W_{n-1}} \to \mathrm{Kerf_{n-2}} \to 0$, and so on, it is not difficult to obtain that W_i is w-injective $(i = 1, 2, \ldots, n)$. Meanwhile. According to our assumption and lemma 4.1, we can get W_i is faithful.

(4) $\Rightarrow$ (1) Assume that there exists an exact sequence

$$0 \to K \to W_n \xrightarrow{f_n} \cdots \to W_2 \xrightarrow{f_2} W_1 \xrightarrow{f_1} E \to 0$$

with $W_i \in w$ and $\mathrm{Kerf_1} \in w^{\perp}$, W_1 is faithful and W_i is w-injective $(i = 1, 2, \ldots, n)$, where E is an injective cogenerator of R-Mod. Consider the short exact sequence $0 \to \mathrm{Kerf_1} \to \mathrm{W_1} \to \mathrm{E} \to 0$, since $\mathrm{Kerf_1} \in w^{\perp}$, so $W_1 \to E$ is a w-precover of R. Note that W_1 is w-injective and faithful, and $W_1 \to E$ is surjective, we can obtain that w is 1-cotilting torsion-free class by lemma 4.1. Therefore, w is n-cotilting torsion-free class.

Theorem 4.2 Suppose w is an n-cotilting torsion-free class in R-Mod. If w is closed under direct limits. Then w is an cover class.

Proof It is easy to prove by theorem 2.5 and [18, theorem 2.2.8].

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基于面板协整检验的地区学前教育发展实证研究

王胜青[1], 王逸琼[2]

(1. 陇南师范高等专科学校 数信学院, 甘肃 陇南 742500)
(2. 天水市卫生学校 信息技术教育处, 甘肃 天水 741000)

摘 要: 基于陇南市各县区 2003–2016 年的动态面板数据, 利用经济计量预测分析软件 Eviews8.0 建立面板数据协整模型, 实证分析陇南市各县区学前教育在园幼儿数、专任教师数、校舍面积数和当地经济发展水平之间的协整关系. 研究结果表明: 在园学生数与专任教师数、校舍面积数、当地人均 GDP 在统计上存在长期均衡关系. 在保持其他变量不变的情况下, 专任教师数每增加 1%, 在园学生数平均增加 0.093348%; 校舍面积数每增加 1%, 在园学生数平均增加 0.158417%; 人均 GDP 每增加 1%, 在园学生数平均增加 0.604851%. 从教育经济和办学指标要求看, 这种长期均衡关系有其解释意义. 影响陇南学前教育发展规模的主要因素是地区经济发展水平, 充足的校舍面积可以满足日益增多的学龄儿童入园, 配备足额的专任教师可以保证学前教育的发展质量.

关键词: 面板数据; 协整检验; 学前教育; 发展水平

近年来, 随着我国经济的飞速发展和人民生活水平的不断提高, 在义务教育已基本普及之后, 学前教育已成为国家重视、民众关心的重大教育问题之一. 2010 年 11 月, 为解决民众关切的 "入园难、入园贵" 问题, 国务院颁布了《关于当前发展学前教育的若干意见》(以下简称 "国十条"), 并引导各地实施了两轮学前教育三年行动计划, 强力推动学前教育发展. 据统计, 截至 2016 年底, 我国学前教育毛入园率达到 77.4%, 普惠性幼儿园占比上升到 60%左右, "入园难、入园贵" 问题得到有效缓解. 整体而言, 短期内政府的强力推进和中央财政的一系列支持政策, 带动地方和社会学前教育投入大幅增长, 无疑让我国学前教育资源得到了迅速扩大, 但因为学前教育的公共产品特性和地区的差异性, 仍需要政府长期兑现 "政府主导、社会参与"、"地方为主、中央奖补" 的投入体制政策, 长远构建 "广覆盖、保基本、有质量的学前教育公共服务体系". 因此, 从宏观上系统研究学前教育发展水平的影响因素及其长期均衡关系, 对政府进行学前教育资源配置、保障学前教育均衡发展具有十分重要的意义和价值.

目前, 我国学前教育研究已逐步成为受关注的教育研究分支领域, 研究范围不断拓展, 研究内容不断丰富, 研究成果的应用性与实效性明显增强. 关于学前教育发展水平的影响因素, 学者普遍认为: 一个地区学前教育发展水平是由该地区对学前教育的供给和需求水平共同决定的. 从供给角度来看, 一个地区的经济发展水平会影响地区政府的财政支出能力及可用

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于教育的资源数量, 进而影响该地区教育事业的发展水平, 同时地区政府对于学前教育事业的努力程度及对学前教育的投入水平也会影响该地区学前教育的发展; 另外, 一个地区的人口密度也会影响学前教育的供给, 人口越稀疏的地区, 学前教育的规模效应越难以发挥, 供给的单位成本会越高, 导致供给数量相应减少. 从需求角度来看, 一个地区的经济发展水平及人口结构因素会影响对学前教育的需求量. 经济发展水平越高的地区, 其居民的总体支付能力也就越高, 对学前教育的需求量也会越大; 而地区人口中适龄儿童的数量越多, 对于学前教育的竞争会更加激烈, 在一定的学前教育供给水平下, 学前教育机会会相应减少. 在实证研究方面, 张雪、袁连生、田志磊 (2012) 利用 1996-2009 年的省级面板数据, 分析了地区学前教育发展水平的影响因素. 研究发现, 与经济发展水平和人口结构因素相比, 当前政府学前教育财政投入水平对地区学前教育发展的影响相对较小, 且在经济意义上并不显著. 通过将全国地区分为东部、中部和西部进行分组分析表明, 政府对学前教育的财政投入在东部、中部、西部及少数民族自治区对学前教育发展的作用各异. 在我国现行学前教育财政体制下, 政府投入如果主要用于公办园质量的提高, 则其扩大学前教育机会的作用会被抑制 [1]. 唐吉洪、王雪标等 (2014) 以幼儿园数为因变量利用辽宁省 2000-2012 年的相关统计数据建立动态面板模型分析了学前教育发展与教育财政投入、城镇化水平以及地区经济发展水平的相依关系. 统计分析结果表明：地区教育财政投入对学前教育的发展并没有统计显著影响, 而经济发展水平、城镇化水平对学前教育的发展影响显著. 政府在促进城镇化的同时也有效促进了学前教育水平的提高 [2]. 叶平枝、张彩云 (2015) 基于国家 2010—2012 年公布的相关数据, 对影响发达地区学前教育发展的各类因素进行因子分析和区域比较. 结果发现, 发达地区学前教育发展的影响因素可分为四个方面：质量因子、机会和投入因子、物质条件因子、负向影响因子, 其中质量因子影响最大; 发达地区学前教育的发展与其经济发展并不完全同步, 上海、北京学前教育的发展与其经济发展较为匹配, 广东、浙江的一些指标却低于全国平均水平; 发达地区学前教育城乡发展不够均衡, 上海、北京城乡差异较小, 广东城乡差异显著, 主要表现在生师比、教师学历和职称等方面; 发达地区幼儿园教师未评职称的比例较高, 普遍达到 50%. 建议增强政策的系统性, 增加并优化学前教育投入, 设立幼儿园教师职称晋升系列, 促进城乡学前教育均衡发展 [3]. 宋佳、吴钰洁 (2017) 运用我国 2003-2014 年时间序列数据, 运用多元逐步回归法, 探讨了影响我国学前教育发展的主要影响因素. 研究发现, 除市场化发展程度外, 政府支持力度、人口结构、家庭对教育重视程度、社会发展水平以及经济发展水平均对我国学前教育发展有显著的促进作用, 也为我国民办学前教育快速发展和我国学前教育综合发展较慢提供了一种较为合理的解释 [4]. 王胜青 (2018) 选取陇南市 2002-2016 年的统计数据, 在协整分析的基础上建立学前教育发展规模影响因素的向量自回归模型 (VAR), 然后综合运用格兰杰 (Granger) 因果检验、脉冲响应函数和方差分解等研究方法, 对经济发展水平、幼儿园校舍面积、专任教师数等因素影响学前教育发展规模的效应进行实证分析. 研究结果表明: 从长期看, 陇南市学前教育发展规模与经济发展水平、幼儿园校舍面积、专任教师数之间存在长期均衡的协整关系 [5].

基于上述研究成果, 本文以秦巴山区集中连片贫困地区陇南市八县一区的数据为样本, 通过建立面板数据模型, 实证研究在园幼儿数、专任教师数、校舍面积数和当地经济发展水平等相关因素的协整关系, 分析学前教育发展水平的影响因素, 以期使政府财政投入、家庭

成本分担能更有效地促进学前教育发展. 本文研究思路的创新点是, 将政策财政投入的效果指标用校舍面积数和专任教师数来反映, 家庭成本分担的基本保证用当地经济发展水平 (人均 GDP) 来体现. 对于一个地区或一所幼儿园, 其在园幼儿数、校舍面积数、专任教师数、人均 GDP 等指标应当是一个长期均衡发展的系统, 任何一个指标过度增长都会造成资源浪费, 所以在发展过程中要保持系统的稳定.

1 研究方法及变量选取

本文研究方法选择的是面板数据建模, 运用协整理论进行分析, 利用经济计量预测分析软件 Eviews8.0 进行实现. 面板数据模型的定义为 [6]:

设有被解释变量 Y_{it} 与 $k\times 1$ 维解释变量向量 $X_{it}=(x_{1,it},x_{2,it},\cdots,x_{k,it})'$, 满足多元线性关系:

$$Y_{it}=\alpha_{it}+X_{it}'\beta_{it}+u_{it}\ ,\ i=1,2,\cdots,N;t=1,2,\cdots,T \tag{1}$$

其中 N 表示截面成员个数,T 表示每个截面成员观测时期的总数, 参数 α_{it} 表示模型的常数项,β_{it} 表示对应于解释变量向量 X_{it} 的 $k\times 1$ 维系数向量, k 表示解释变量个数. 随机误差项 u_{it} 相互独立, 且满足零均值、等方差为 σ_{it}^2 的假设, 则式 (1) 为面板数据模型.

为了避免 "伪回归" 问题, 面板数据在进行协整分析之前应先进行变量的平稳性检验, 有效的方法是面板单位根检验. 经检验, 若变量序列都平稳, 可直接进行回归分析; 若不平稳, 看一阶差分是否平稳. 如果原始时间序列不平稳, 而经过一阶差分变成平稳的, 原始序列就是一阶单整, 即为 I(1) 过程. 当各变量均不平稳, 而经过一阶差分变成平稳时, 可进行协整检验和协整回归 [7].

考虑到数据的可获得性和研究需要, 文章选取陇南市 8 县 1 区 2003–2016 年学前教育在园学生人数 (XS)、专任教师数 (JS)、校舍面积 (MJ)、人均 GDP(GDP) 等作为研究变量. 同时, 为消除物价变动对 GDP 的影响, 采用陇南市各年度居民消费价格指数对各县区人均 GDP 数据进行平减, 将样本期内人均 GDP 数据调整为上年的不变价格. 为消除数据中存在异方差问题, 对变量 XS、JS、MJ、GDP 分别取自然对数, 使变量变为弹性变量, 并记为 LNXS、LNJS、LNMJ、LNGDP, 其中 LNXS 为被解释变量, 其余为解释变量. 模型中相关变量的统计性描述见表 1.

2 实证分析

2.1 面板单位根检验

平稳性检验是时间序列建模分析的一个重要内容, 对于一些非平稳序列, 如果不是同阶单整或不存在协整关系, 统计回归得出的结果可能是伪回归. 因此, 为了避免伪回归, 需要对时间序列进行单位根检验以判断其平稳性. 面板数据的单位根检验主要有 LLC、Breitung、IPS、ADF-Fisher、PP-Fisher 等 5 种方法, 前两种方法适用于同质面板单位根检验, 后三种方法适用于异质面板单位根检验. 吕延方、陈磊 (2010) 比较详细地介绍了面板单位根检验方法及其稳定性 [8]. 为了保证检验结果统计稳健, 同时运用上述 5 种面板数据单位根检验方法对模型中变量的序列平稳性进行检验, 变量 LNXS、LNJS、LNMJ、LNGDP 含个体截距项和趋势项

检验结果分别见表 2-5.

表 1 各变量的描述性统计

	LNXS	LNJS	LNMJ	LNGDP
均值	7.984092	4.144750	8.514025	8.579547
中位数	7.999337	4.110874	8.403906	8.608223
最大值	10.25048	5.986452	11.14610	10.00797
最小值	6.169611	2.833213	4.969813	6.914581
标准差	0.910975	0.727691	1.101849	0.764414
偏度	0.077129	−0.021502	0.160994	−0.109709
峰度	2.701366	2.207962	2.959499	2.206958
JB 统计量	0.593133	3.303159	0.552910	3.554561
P 值	0.743366	0.191747	0.758468	0.169097
观测数	126	126	126	126
截面数	9	9	9	9

表 2 LNXS 及其一阶差分面板单位根检验

	LNXS		D(LNXS)	
Method	Statistic	Prob.**	Statistic	Prob.**
Null:Unit root(assumes common unit root process)				
Levin,lin & chut	−1.67451	0.0470	−5.81483	0.0000
Breitung t-stat	1.73815	0.9589	−3.83484	0.0001
Null:Unit root(assumes individual unit root process)				
Im,Pesaran & shin w-stat	1.07901	0.8597	−4.80037	0.0000
ADF-Fisher chi-square	13.0227	0.7902	53.5662	0.0000
PP-Fisher chi-square	11.5946	0.8674	86.6565	0.0000

表 3 LNJS 及其一阶差分面板单位根检验

	LNJS		D(LNJS)	
Method	Statistic	Prob.**	Statistic	Prob.**
Null:Unit root(assumes common unit root process)				
Levin,lin & chut	−1.44817	0.0738	−7.36587	0.0000
Breitung t-stat	3.65420	0.9999	−4.36518	0.0000
Null:Unit root(assumes individual unit root process)				
Im,Pesaran & shin w-stat	1.61538	0.9469	−4.36919	0.0000
ADF-Fisher chi-square	13.9870	0.7299	49.7019	0.0001
PP-Fisher chi-square	16.8277	0.5350	90.0618	0.0000

表 4 LNMJ 及其一阶差分面板单位根检验

	LNMJ		D(LNMJ)	
Method	Statistic	Prob.**	Statistic	Prob.**
Null:Unit root(assumes common unit root process)				
Levin,lin & chut	0.03481	0.5139	–10.5111	0.0000
Breitung t-stat	4.13884	1.0000	–6.42896	0.0000
Null:Unit root(assumes individual unit root process)				
Im,Pesaran & shin w-stat	3.70120	0.9999	–6.15578	0.0000
ADF-Fisher chi-square	4.60417	0.9994	64.7513	0.0000
PP-Fisher chi-square	4.01839	0.9998	99.3019	0.0000

表 5 LNGDP 及其一阶差分面板单位根检验

	LNGDP		D(LNGDP)	
Method	Statistic	Prob.**	Statistic	Prob.**
Null:Unit root(assumes common unit root process)				
Levin,lin & chut	–0.52588	0.2995	–7.21046	0.0000
Breitung t-stat	1.58624	0.9437	–4.84635	0.0000
Null:Unit root(assumes individual unit root process)				
Im,Pesaran & shin w-stat	0.91631	0.8202	–3.72649	0.0001
ADF-Fisher chi-square	15.5596	0.6233	43.5799	0.0007
PP-Fisher chi-square	11.8469	0.8550	63.8393	0.0001

从表 2–5 可知, 对于各个变量的水平值进行检验时, 均不能拒绝 “存在单位根” 的原假设, 即各变量均是非平稳过程. 而对各变量的一阶差分值进行检验时, 检验结果均在 1%显著水平上拒绝了原假设, 即各变量的一阶差分时间序列为平稳过程. 因此, 四个时间序列变量均为一阶单整 I(1) 过程, 从理论上讲可以进行协整检验和协整回归.

2.2 面板协整检验

假设同期截面独立, 基于残差的面板协整检验方法分别选取 Kao(1999) 检验和 Pedroni (1999) 检验.

1) Kao 检验

Kao(1999) 检验是针对同质面板的协整检验, 用 Eviews8.0 检验时只有一种协整统计量 ADF, 并且只含个体截矩项一种形式 (Individual intercept), ADF 检验结果为: $t = -3.203779$, $p = 0.0007$. 说明检验统计量 ADF 的 p 值小于给定的 0.05 显著性水平, 从而拒绝没有协整关系的原假设, 说明对所有个体而言, 变量 Lnxs、lnjs、lnmj、lngdp 之间存在协整关系.

2) Pedroni 检验

Pedroni (1999) 是针对异质面板的协整检验, 有 7 种协整统计量, 分别为: panel v-statistic、panel rho-statistic、panel pp-statistic、panel ADF-statistic、Group rho-statistic、Group pp-statistic、Group ADF-statistic, 其中前 4 个是组内统计量 (Within-dimension), 后 3 个是组间

统计量 (Between-dimension)[9]. 选择既有截矩项又有趋势项进行检验, 检验结果见表 6.

表 6 Pedroni 协整检验

	Statistic	Prob.
Panel v-Statistic	−1.889534	0.9706
Panel rho-Statistic	1.455362	0.9272
Panel PP-Statistic	−3.061034	0.0011
Panel ADF-Statistic	−2.951997	0.0016
Group rho-Statistic	2.885537	0.9980
Group PP-Statistic	−4.862054	0.0000
Group ADF-Statistic	−2.642061	0.0041

从表 6 知, 组内统计量 Panel PP-statistic、Panel ADF-statistic 和组间统计量 Group PP-statistic、Group ADF-statistic 的 P 值小于给定的 0.05 显著性水平, 另外三个不显著, 从大多数统计量而言, 检验均拒绝了无协关系的原假设, 按异质面板协整检验的说法, 说明对一部分个体而言这些变量之间存在协整关系. 同理, 只有截矩项或 NO 的检验结果同上, 说明变量 Lnxs、lnjs、lnmj、lngdp 之间存在协整关系.

2.3 模型设定检验

建立面板数据模型时, 在统计学意义下先要检验确定模型的类型. 首先进行 chow 检验确定其维度, 此时对应的原假设和备择假设为:

$$\text{Cross}-\text{section FE test}:\begin{cases}\mathrm{H}_0:\text{个体Pool模型, 时点 FE 模型.}\\ \mathrm{H}_1:\text{个体, 时点 FE 模型.}\end{cases}$$

$$\text{Period FE test}:\begin{cases}\mathrm{H}_0:\text{个体 FE 模型, 时点 Pool 模型.}\\ \mathrm{H}_1:\text{个体, 时点 FE 模型.}\end{cases}$$

$$\text{Cross - section and Period test}:\begin{cases}\mathrm{H}_0:\text{Pool模型}\\ \mathrm{H}_1:\text{个体, 时点 FE 模型.}\end{cases}$$

以变量 lnxs 为被解释变量, 变量 lnjs、Lnmj、Lngdp 为解释变量进行个体和时点双固定效应面板数据回归, 对回归结果进行冗余固定效应和似然比检验, 检验结果见图 1.

Effects Test	Statistic	d.f.	Prob.
Cross-section F	27.318587	(8,101)	0.0000
Cross-section Chi-square	145.125435	8	0.0000
Period F	4.386406	(13,101)	0.0000
Period Chi-square	56.400347	13	0.0000
Cross-Section/Period F	12.665815	(21,101)	0.0000
Cross-Section/Period Chi-square	162.564270	21	0.0000

图 1 冗余固定效应和似然比检验

从图 1 可知, 个体和时点的 F 和卡方检验的 P 值均小于 0.05 的显著性水平, 都拒绝了原假设, 说明此模型至少不是 Pool 模型.

其次基于个体和时点双随机效应进行回归, 对回归结果进行 Hausman(豪斯曼) 检验, 发现检验统计量的卡方统计值均为 0, 说明检验的方差是非有效的, 双随机效应检验有问题.

然后, 基于个体固定效应、时点随机效应进行回归, 对回归结果进行 Hausman(豪斯曼) 检验, 发现检验结果拒绝了原假设, 说明时点不是随机效应; 同样, 基于个体随机效应、时点固定效应进行回归, 对回归结果进行 Hausman(豪斯曼) 检验, 发现检验结果拒绝了原假设, 说明个体也不是随机效应. 检验结果分别见表 7.

表 7 个体或时点单个随机效应的 Hausman 检验

Test Summary	Chi-Sq. Statistic	Chi-Sq. d.f.	Prob.
Period random	27.461229	3	0.0000
Cross-section random	61.025686	3	0.0000

2.4 协整回归与模型解释

通过上述检验, 模型设定中既排除了 Pool 模型, 又排除了个体或时点为随机效应模型. 因此, 为控制不可观测因素对学前教育发展的影响, 该模型选择个体固定效应、时点不变的 FGLS 估计, 并用既有个体异方差又有同期截面相关设置权重, 进行面板协整回归, 协整回归方程为:

$$\ln\hat{xs} = 1.059074 + 0.093348 * \ln\hat{js} + 0.158417 * \ln\hat{mj} + 0.604851 * \ln\hat{gdp} \tag{2}$$

从图 2 可知, 截矩项 C 及变量 lnjs、lnmj、lngdp 在给定显著性水平 0.05 下均显著, 回归系数也符合经济学和教育学意义.

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	1.059074	0.203408	5.206644	0.0000
LNJS?	0.093348	0.034747	2.686479	0.0083
LNMJ?	0.158417	0.018503	8.561469	0.0000
LNGDP?	0.604851	0.023566	25.66588	0.0000
Fixed Effects (Cross)				
_WD--C	0.493435			
_CX--C	-0.156908			
_WX--C	-0.048447			
_DC--C	0.119086			
_KX--C	-0.093962			
_XH--C	0.669900			
_LX--C	0.788429			
_HX--C	-0.396073			
_LD--C	-1.375460			

图 2 个体固定效应的协整回归

说明陇南市在园学生数与专任教师数、校舍面积数、当地人均 GDP 在统计上存在长期均衡关系, 而且解释变量与被解释变量的相依关系与预期一致, 可决系数为 0.9715, 修正后的可决系数为 0.9688. 式 (2) 表明, 在保持其他变量不变的情况下, 专任教师数每增加 1%, 在园学生数平均增加 0.093348%; 校舍面积数每增加 1%, 在园学生数平均增加 0.158417%; 人均 GDP 每增加 1%, 在园学生数平均增加 0.604851%. 而且从教育经济和办学指标要求看, 这种长期均衡关系有其解释意义. 即: 影响学前教育发展水平的主要因素是地区经济发展水平, 其次是校舍面积数和专任教师数.

通过固定效应发现, 不同个体对学前教育的发展存在差异, 就陇南市而言, 武都区、宕昌县、西和县、礼县等个体的不可观测因素对学前教育的发展具有正向作用, 而其它 5 县个体效应为负 (见表 8).

2.5 误差修正模型

从图 2 可知, 在园学生数与专任教师数、校舍面积数、人均 GDP 存在长期均衡关系, 但如果短期内偏离这种长期均衡关系, 它们又是如何修回的? 此时需要生成误差修正模型.

表 8 协整回归方程个体固定效应

地名	武都区	成县	文县	宕昌	康县
效应值	0.493435	−0.156908	−0.048447	0.119086	−0.093962
地名	西和县	礼县	徽县	两当县	
效应值	0.669900	0.788429	−0.396073	−1.375460	

首先生成残差序列 ec, 求得滞后一阶序列 ec(-1), 然后生成各变量的一阶差分序列 dlnxs、dlnjs、dlnmj、dlngdp, 选择个体固定效应、时点不变的 FGLS 估计, 用既有个体异方差又有同期截面相关设置权重, 生成如下误差修正模型.

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.171210	0.013469	12.71183	0.0000
DLNJS?	0.138371	0.024595	5.626101	0.0000
DLNMJ?	0.045902	0.011141	4.119977	0.0001
DLNGDP?	-0.367064	0.069822	-5.257170	0.0000
EC?(-1)	-0.276509	0.070969	-3.896222	0.0002
Fixed Effects (Cross)				
_WD--C	0.063220			
_CX--C	-0.042445			
_WX--C	0.040457			
_DC--C	0.071143			
_KX--C	0.015670			
_XH--C	0.001053			
_LX--C	-0.042702			
_HX--C	-0.037015			
_LD--C	-0.069380			

图 3 误差修正模型

从图 3 可知, 滞后 1 期的各变量均通过了显著性水平为 0.05 的 P 值检验, 误差修正模型回归方程为:

$$d\ln \hat{xs} = 0.171210 + 0.138371 * d\ln \hat{js} + 0.045902 * d\ln \hat{mj} - 0.367064 * d\ln \hat{gdp} - 0.276509ec(-1) \tag{3}$$

式 (3) 中 EC (-1) 的系数 -0.276509, 是短期内如果偏离长期均衡关系时的修复力度. 各地个体固定效应值见表 9.

表 9 误差修正模型个体固定效应

地名	武都区	成县	文县	宕昌	康县
效应值	0.063220	−0.042445	0.040457	0.071143	0.015670
地名	西和县	礼县	徽县	两当县	
效应值	0.001053	−0.042702	−0.037015	−0.069380	

3 结论及建议

文章基于陇南市各县区 2003--2016 年的面板数据建立模型, 实证分析陇南市学前教育在

园幼儿数、专任教师数、校舍面积数和当地经济发展水平之间的协整关系. 从各序列数据分布图可知, 各县区在园学生数、专任教师数、校舍面积数在 2003–2010 年期间无明显上升趋势, 但 2011–2016 年的 6 年间有显著的上升趋势. 说明实施两轮三年行动计划的政策驱动对学前教育发展有显著的影响. 各县区 2003–2016 年人均 GDP 呈现逐年上升趋势, 在时间点 2008 年之后有明显的上升趋势, 这与灾后重建的政策驱动有关, 而与两轮次的学前教育三年行动计划无关.

从协整回归模型可知, 在园学生数与专任教师数、校舍面积数、当地人均 GDP 在统计上存在长期均衡关系. 在保持其他变量不变的情况下, 专任教师数每增加 1%, 在园学生数平均增加 0.093348%; 校舍面积数每增加 1%, 在园学生数平均增加 0.158417%; 人均 GDP 每增加 1%, 在园学生数平均增加 0.604851%. 从教育经济和办学指标要求看, 这种长期均衡关系有其解释意义: 影响陇南学前教育发展水平的主要因素是地区经济发展水平, 因为从需求角度看, 经济发展水平越高的地区, 其居民的支付能力也就越高, 对学前教育的需求也会越大; 其次是校舍面积数, 充足的校舍面积可以满足日益增多的学龄儿童全部入园, 可以促进入园率的提高; 再次是专任教师数, 尽管专任教师数的边际贡献不是很大, 但从学前教育质量提高而言, 配备足额的专任教师可以保证学前教育的发展质量.

综上所述, 当政府完成了学前教育两轮三年行动计划之后, 陇南市各县区幼儿园校舍面积得到了快速增长, 专任教师得到一定的补充, 学前教育三年毛入园率平均达到了 92.12%, 学前教育规模基本趋于稳定, 全市各县区学前教育发展规模与专任教师数、幼儿园校舍面积数、地方经济发展水平之间从长远看将会保持一种长期均衡关系. 但由于陇南属秦巴山区集中连片贫困区, 是甘肃省贫困人口最多的地区之一, 经济落后、山大沟深、乡村农户居住分散、村级幼儿园短缺仍是制约学前教育发展的主要因素. 在短期内, 这种均衡关系会被打破, 式 (3) 给出了短期内如果偏离长期均衡关系时的修复力度. 今后, 各级地方政府还是要从发展地方经济抓起, 通过提高学前教育投入和居民支付能力来增加对学前教育的需求, 扩大学前教育规模. 同时, 针对各县区及城乡经济水平差异大、学前教育发展不均衡的问题, 地方政府应制订统筹城乡教育发展规划, 宏观调控城乡教育资源的配置. 一方面, 要加大对农村园、薄弱园的扶持力度, 充分发挥经济为教育服务的功能, 公共财政经费向农村及乡镇学前教育倾斜, 重点是建立长效、规范的保障制度, 用优质的教学条件和学前教育服务来留住幼儿; 另一方面, 随着城镇化进程的推进, 对于县城学前教育, 面对进城务工人员随迁子女人数剧增的趋势, 要科学测算适龄儿童, 积极扩大学前教育资源量, 满足随迁儿童的入园需求. 在保证适龄儿童有园上的同时, 还要着力解决上好园的问题.

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Empirical Study on the Development of Preschool Education in Area Based on Panel Co-Integration Test

WANG Sheng-qing[1], WANG Yi-qiong[2]

(1. College of Mathematics and Information Technology, Longnan Teachers College, Longnan 742500, China)
(2. Information Technology Education Department, Tianshui Health School, Tianshui 741000, China)

Abstract: Based on dynamic panel data from 2003 to 2016, this paper applies panel co-integration model by Eviews 8.0 to analyze the co-integration relationship among numbers of children, full-time teachers, building area and local economic development level of preschool education of every districts of Longnan city. The results show that there is long-term equilibrium relation among the numbers of children in kindergarten, full-time teachers, building area and local per capita GDP. When other variables remain unchanged, each increment of one percent in the numbers of full-time teachers results in an increase of 0.093348% for the numbers of children; the numbers of building area ascend one percent and the numbers of children increases 0.158417%; per capita GDP increases one percent and the numbers of children is raise 0.604851%. According to the education economy and schooling index, this long-term equilibrium relation exhibits explain meaning. These results indicate that the main factors influence the development scale of Longnan preschool education is local economic development level, ample building area can meet the demands of increasing school children, and amount of full-time teachers can ensure the development quality of preschool education.

Keywords: panel data; co-integration test; preschool education; development level

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XIBEI SHIFAN DAXUE XUEBAO

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一类分数阶广义捕食者-食饵模型的动力学分析

蒲武军，杜争光
（陇南师范高等专科学校 数学系，甘肃 陇南 742500）

摘要：讨论一类分数阶广义捕食者-食饵模型. 通过定性分析方法研究了该模型解的存在唯一性、非负性和有界性，并利用分数阶系统的稳定性理论给出了该系统正平衡点的局部渐近稳定和全局渐近稳定的充分条件. 数值模拟表明，系统的参数和阶数不仅影响系统平衡点的收敛速度，也对系统的稳定性产生很重要的影响.

关键词：分数阶微分方程；广义捕食者-食饵模型；有界性；稳定性
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Dynamical analysis of a fractional-order generalist predator-prey model

PU Wu-jun，DU Zheng-guang
(Department of Mathematics，Longnan Teachers College，Longnan 742500，Gansu，China)

Abstract：A fractional-order generalist predator-prey model is investigated. By using qualitative analysis，the existence，uniqueness，non-negativity and boundedness of all solutions are studied. Then，applying the stability theory of fractional-order system，some sufficient conditions for the locally asymptotical stability and globally asymptotic stability of the equilibriums are obtained. Moreover，some numerical simulations are presented to demonstrate that the parameter and order of the system not only affect the convergence rate of the equilibriums，but also have an important influence on the stability of the system.
Key words：fractional-order differential equation；generalist predator-prey model；boundedness；stability

分数阶微积分将导数和积分推广至任意阶，分数阶微分方程常被用来描述黏弹性、电动电路、电化学现象等. 近年来，随着分数阶微积分方程研究的不断发展，分数阶种群系统的动力学研究逐渐成为了一个热点课题，并且已经取得了一系列好的结果[1-4]. Erbach 等[5]讨论了一类广义捕食系统的双稳态和极限环的存在性问题，指出广义捕食系统的动力学行为较具有一般反应函数的捕食系统更加丰富. El-Shahed[6]等讨论了一类分数阶广义捕食者-食饵模型正平衡点的存在性、稳定性和极限环. 目前，针对 Holling-Ⅱ型功能反应的分数阶捕食系统的研究已有许多结果[7-9].

文中考虑一类具有 Holling-Ⅱ型功能反应的分数阶广义捕食者-食饵模型：

$$\begin{cases} \mathrm{D}^{\alpha}x(t)=rx(t)\left(1-\dfrac{x(t)}{K}\right)- \\ \qquad \dfrac{\beta x(t)y(t)}{1+\sigma x(t)}, \\ \mathrm{D}^{\alpha}y(t)=\dfrac{\delta x(t)y(t)}{1+\sigma x(t)}+ \\ \qquad \dfrac{cy(t)}{1+dy(t)}-ey(t), \end{cases} \tag{1}$$

其中 $\alpha\in(0,1]$，$r,K,\beta,\delta,\sigma,c,d,e$ 均为正常数，x,y 分别表示食饵和捕食者的种群密度，初始条

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作者简介：蒲武军（1979—），男，甘肃庄浪人，讲师，硕士. 主要研究方向为应用微分方程和生物数学.
E-mail：puwj2005@163.com

件为 $x(0)>0, y(0)>0$ 且 $c>e$. 有关系统(1)的详细生物学意义参见文献[5].

1　预备知识和引理

首先介绍分数阶微积分的相关知识.

定义 1[10]　设 f: $\mathbb{R}^+ \longrightarrow \mathbb{R}$，则其 Riemann-Liouville 分数阶积分定义如下：

$$\mathrm{I}^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s,$$

其中 $\Gamma(\cdot)$ 是伽玛函数.

定义 2[11]　Caputo 分数阶导数定义如下：

$$\mathrm{D}^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\mathrm{d}s,$$

其中 $\alpha>0, n-1<\alpha<n$，n 是整数，$\Gamma(\cdot)$ 是伽玛函数.

设分数阶系统

$$\begin{cases}\mathrm{D}^\alpha x(t)=f_1(x,y),\\ \mathrm{D}^\alpha y(t)=f_2(x,y),\\ x(0)=x_0, y(0)=y_0,\ \alpha\in(0,1]\end{cases} \tag{2}$$

有平衡点 (x_e, y_e).

引理 1[12]　分数阶系统(2)的平衡点 (x_e, y_e) 是局部渐近稳定的当且仅当雅克比矩阵

$$J=\begin{pmatrix}\partial f_1/\partial x & \partial f_1/\partial y\\ \partial f_2/\partial x & \partial f_2/\partial y\end{pmatrix}$$

在平衡点 (x_e, y_e) 的所有特征值 λ_i 满足条件 $|\arg(\lambda_i)|>\frac{\alpha\pi}{2}$.

引理 2[13]　设 $x(t)\in\mathbb{R}^+$ 是一个连续可微函数，则对任意时刻 $t\geqslant t_0$，有

$$\mathrm{D}^\alpha\left\{x(t)-x^*-x^*\ln\frac{x(t)}{x^*}\right\}\leqslant \left\{1-\frac{x^*}{x(t)}\right\}\mathrm{D}^\alpha x(t),$$

$$x^*\in\mathbb{R}^+, \forall\alpha\in(0,1).$$

引理 3[14]　考虑如下分数阶系统：

$$\mathrm{D}^\alpha x(t)=f(t,x),\ t>t_0, \tag{3}$$

其中 $x(0)=x(t_0), \alpha\in(0,1], f:[t_0,\infty)\times\Omega\longrightarrow\mathbb{R}^n, \Omega\subset\mathbb{R}^n$. 如果 $f(t,x)$ 关于 x 满足局部利普希茨条件，则系统(3)在 $[t_0,\infty)\times\Omega$ 上存在唯一解.

引理 4[14]（比较定理）　若 $\mathrm{D}^\alpha x(t)\leqslant\mathrm{D}^\alpha y(t)$，$x(0)=y(0)$，其中 $\alpha\in(0,1]$，则 $x(t)\leqslant y(t)$.

引理 5[15]　设 $u(t)$ 是定义在 $[t_0,\infty)$ 上的连续函数且满足

$$\begin{cases}\mathrm{D}^\alpha u(t)\leqslant-\lambda u(t)+\mu,\\ u(t_0)=u_{t_0},\end{cases}$$

其中 $0<\alpha<1, (\lambda,\mu)\in\mathbb{R}^2$ 且 $\lambda\neq0$，初始时刻 $t_0\geqslant0$. 则

$$u(t)\leqslant\left(u_{t_0}-\frac{\mu}{\lambda}\right)E_\alpha[-\lambda(t-t_0)^\alpha]+\frac{\mu}{\lambda},$$

其中 E_α 是 M-L 函数，即

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)},\ z\in\mathbb{C}.$$

2　主要结果及其证明

2.1　解的存在唯一性

定理 1　对任意初值 $(x(t_0), y(t_0))\in\Omega$，$\Omega=\{(x,y)\in\mathbb{R}^2: \max\{|x|,|y|\}\leqslant M\}$，系统(1)存在唯一解 $X=(x,y)\in\Omega$.

证明　令 $X=(x,y), \bar{X}=(\bar{x},\bar{y})$，定义映射 $H(X)=(H_1(X), H_2(X))$，其中

$$\begin{cases}H_1(X)=rx\left(1-\dfrac{x}{K}\right)-\dfrac{\beta xy}{1+\sigma x},\\[2ex] H_2(X)=\dfrac{\delta xy}{1+\sigma x}+\dfrac{cy}{1+dy}-ey.\end{cases} \tag{4}$$

对 $\forall X, \bar{X}\in\Omega$，由(4)式可得

$$\|H(X)-H(\bar{X})\|=\sum_{i=1}^{2}|H_i(X)-H_i(\bar{X})|\leqslant \left[r+\left(\frac{2r}{K}+\beta+\delta\right)M\right]|x-\bar{x}|+ \left[(\beta+\delta)\left(M+\frac{1}{\sigma}\right)+c+e\right]|y-\bar{y}|\leqslant L\|X-\bar{X}\|,$$

其中

$$L=\max\left\{r+\left(\frac{2r}{K}+\beta+\delta\right)M, (\beta+\delta)\left(M+\frac{1}{\sigma}\right)+c+e\right\},$$

即系统(1)关于 X 满足局部利普希茨条件，由引理 3 知系统(1)存在唯一解.　】

2.2　非负性和有界性

定理 2　记 $\Omega_+=\{(x,y)\in\Omega: x\in\mathbb{R}_+, y\in\mathbb{R}_+\}$，且 $x(t_0)>0, y(t_0)>0$. 则对任意始于 Ω_+ 的初值，系统(1)的所有解是非负的.

证明　注意到 $x(t_0)>0$，故只需证明对任意的 $t>t_0, x(t)\geqslant0$. 设存在时刻 $t_1>t_0$，使得 $x(t_1)<0$. 由于 $x(t_0)>0$，所以存在 $t_2>t_0$ 使得 $x(t_2)=0$. 记 $\bar{t}=\min\{t_2>0: x(t_2)=0\}$，则当 $t>\bar{t}>t_0$ 时，有

$$D^{\alpha}x(\bar{t})|_{x(\bar{t})=0}=$$
$$rx(\bar{t})\left(1-\frac{x(\bar{t})}{K}-\frac{\beta y(\bar{t})}{1+\sigma x(\bar{t})}\right)=0. \quad (5)$$

另一方面,由 $\bar{t}$ 的定义并结合 $x(t_0)>0$ 知 $x(t)>0,t\in[t_0,\bar{t}]$. 令 $w(t)=-x(t),t\in[t_0,\bar{t})$,则 $w(\bar{t})=0$ 且 $w(t)<0,t\in[t_0,\bar{t})$. 由引理4知 $D^{\alpha}w(t)>0$,即 $D^{\alpha}x(t)<0$,这与(5)式矛盾. 同理可证 $y(t)\geqslant 0$. 】

定理3 集合

$$\Gamma=\left\{(x,y)\in\Omega_+:0\leqslant x+\frac{\beta}{\delta}y\leqslant \frac{K(r+e)^2}{4re}+\frac{\beta c}{\delta de}+\varepsilon,\ \varepsilon>0\right\}$$

是关于系统(1)的正不变集,且对任意初值条件 $(x(t_0),y(t_0))\in\Omega_+$,$\Gamma$ 是关于系统(1)的全局吸引集,并且系统(1)的所有解均有界.

证明 考虑函数 $w(t)=x(t)+\frac{\beta}{\delta}y(t)$,则

$$D^{\alpha}w(t)+ew(t)\leqslant -\frac{rx^2}{K}+(r+e)x+\frac{\beta c}{\delta d}\leqslant \frac{K(r+e)^2}{4r}+\frac{\beta c}{\delta d}.$$

由引理5可得

$$0\leqslant w(t)\leqslant\left(w(t_0)-\frac{K(r+e)^2}{4re}-\frac{\beta c}{\delta de}\right)\times E_{\alpha}[-e(t-t_0)^{\alpha}]+\frac{K(r+e)^2}{4re}+\frac{\beta c}{\delta de}. \quad (6)$$

因此,系统(1)始于 Ω_+ 的所有解均位于集合

$$\Gamma=\left\{(x,y)\in\Omega_+:x+\frac{\beta}{\delta}y\leqslant \frac{K(r+e)^2}{4re}+\frac{\beta c}{\delta de}+\varepsilon,\ \varepsilon>0\right\}$$

中,即 Γ 是关于系统(1)的正不变集.

根据(6)式,若 $w(t_0)\leqslant\frac{K(r+e)^2}{4re}+\frac{\beta c}{\delta de}$,则系统(1)的解停留在集合 Γ 内,$E_{\alpha}[-e(t-t_0)^{\alpha}]\to 0(t\to+\infty)$. 若 $w(t_0)>\frac{K(r+e)^2}{4re}+\frac{\beta c}{\delta de}$,则系统(1)的所有解最终收敛到集合 Γ,故 Γ 是一全局吸引集,且对于任意初值条件 $(x(t_0),y(t_0))\in\Omega_+$,系统(1)的所有解均有界. 】

2.3 平衡点的局部稳定性

令 $D^{\alpha}x(t)=0,D^{\alpha}y(t)=0$,易得系统(1)的4个平衡点分别是 $E_0(0,0)$,$E_1(K,0)$,$E_2\left(0,\frac{c-e}{ed}\right)$,$E_3(x_*,y_*)$. 为证明平衡点的稳定性,先给出系统(1)在点$(x,y)$处的雅克比矩阵

$$J(x,y)=\begin{pmatrix} J_{11} & -\frac{\beta y}{1+\sigma x} \\ \frac{\delta y}{(1+\sigma x)^2} & J_{22}\end{pmatrix}, \quad (7)$$

其中,

$$J_{11}=r-\frac{2rx}{K}-\frac{\beta y}{(1+\sigma x)^2},$$
$$J_{22}=\frac{\delta y}{1+\sigma x}+\frac{c}{(1+dy)^2}-e.$$

显然,系统(1)的平凡平衡点 $E_0(0,0)$ 和边界平衡点 $E_1(K,0)$ 均不稳定.

定理4 若 $red<\beta(c-e)$,则系统(1)的边界平衡点 E_2 局部渐近稳定;若 $red>\beta(c-e)$,则系统(1)的边界平衡点 E_2 不稳定.

证明 由雅克比矩阵(7)可知,系统(1)在 $E_2\left(0,\frac{c-e}{ed}\right)$ 点的雅克比矩阵为

$$J\left(0,\frac{c-e}{ed}\right)=\begin{pmatrix} r-\beta L & 0 \\ \delta L & -Lced\end{pmatrix},$$

其中 $L=\frac{c-e}{ed}$. 当 $red<\beta(c-e)$ 时,易得 E_2 的两个特征值分别为

$$\lambda_1=r-\frac{\beta(c-e)}{ed}<0,\ \lambda_2=-\frac{e(c-e)}{c}<0.$$

于是,$\arg(\lambda_1)=\arg(\lambda_2)=\pi$,即 $|\arg(\lambda_1)|>\frac{\alpha\pi}{2}$,$|\arg(\lambda_2)|>\frac{\alpha\pi}{2}$,由引理1知平衡点 E_2 局部渐近稳定. 若 $red>\beta(c-e)$,则 E_2 的两个特征值异号,所以 E_2 不稳定. 】

定理5 若 $e\sigma<\delta,\beta(c-e)<rde$,且下列条件之一成立:

(i) $\mathrm{tr}(J(x_*,y_*))\leqslant 0$;

(ii) $\mathrm{tr}(J(x_*,y_*))>0$,
$$\mathrm{tr}^2(J(x_*,y_*))-4\det(J(x_*,y_*))\leqslant 0,$$
$$|\mathrm{tr}^2(J(x_*,y_*))-4\det(J(x_*,y_*))|^{\frac{1}{2}}>\mathrm{tr}(J(x_*,y_*))\tan\frac{\alpha\pi}{2}.$$

则系统(1)的正平衡点 $E_3(x_*,y_*)$ 局部渐近稳定.

证明　由(7)式得系统(1)在正平衡点 $E_3(x_*,y_*)$ 的雅克比矩阵为

$$J(x_*,y_*)=\begin{pmatrix}-\frac{rx_*}{K}+\frac{\beta\sigma y_* x_*}{(1+\sigma x_*)^2} & -\frac{\beta x_*}{1+\sigma x_*}\\ \frac{\delta y_*}{(1+\sigma x_*)^2} & -\frac{cdy_*}{(1+dy_*)^2}\end{pmatrix},$$

其特征方程为

$$\lambda^2-\mathrm{tr}(J(x_*,y_*))+\det(J(x_*,y_*))=0,$$

故其特征根可表示为

$$\lambda_1=\frac{1}{2}\left\{\mathrm{tr}(J(x_*,y_*))+[\mathrm{tr}^2(J(x_*,y_*))-4\det(J(x_*,y_*))]^{\frac{1}{2}}\right\},\qquad(8)$$

$$\lambda_2=\frac{1}{2}\left\{\mathrm{tr}(J(x_*,y_*))-[\mathrm{tr}^2(J(x_*,y_*))-4\det(J(x_*,y_*))]^{\frac{1}{2}}\right\}.\qquad(9)$$

若条件(i)成立，则有下列 3 种情形：

情形 1　$\mathrm{tr}(J(x_*,y_*))=0$. 此时 λ_1 和 λ_2 是一对共轭复数根，且 $\arg(\lambda_1)=\frac{\pi}{2},\arg(\lambda_2)=-\frac{\pi}{2}$，故 $|\arg(\lambda_1)|>\frac{\alpha\pi}{2},|\arg(\lambda_2)|>\frac{\alpha\pi}{2}$. 由引理 1 知正平衡点 $E_3(x_*,y_*)$ 局部渐近稳定.

情形 2　$\mathrm{tr}(J(x_*,y_*))<0,\mathrm{tr}^2(J(x_*,y_*))-4\det(J(x_*,y_*))\geqslant 0$. 此时 $\lambda_1<0,\lambda_2<0$，于是 $|\arg(\lambda_1)|>\frac{\alpha\pi}{2},|\arg(\lambda_2)|>\frac{\alpha\pi}{2}$. 由引理 1 知正平衡点 $E_3(x_*,y_*)$ 局部渐近稳定.

情形 3　$\mathrm{tr}(J(x_*,y_*))<0,\mathrm{tr}^2(J(x_*,y_*))-4\det(J(x_*,y_*))<0$. 此时 λ_1 和 λ_2 仍是一对共轭复根，且 $\mathrm{Re}(\lambda_1)=\mathrm{Re}(\lambda_2)<0$，故 $|\arg(\lambda_1)|>\frac{\alpha\pi}{2},|\arg(\lambda_2)|>\frac{\alpha\pi}{2}$. 由引理 1 知正平衡点 $E_3(x_*,y_*)$ 局部渐近稳定.

综上可知，在条件(i)下正平衡点 $E_3(x_*,y_*)$ 局部渐近稳定.

若条件(ii)成立，则由特征值表达式(8)和(9)可知：λ_1 和 λ_2 是一对共轭复根，且

$$\mathrm{Im}(\lambda_1)=-\mathrm{Im}(\lambda_2)=[4\det(J(x_*,y_*))-\mathrm{tr}^2(J(x_*,y_*))]^{\frac{1}{2}}>0,$$

$$\mathrm{Re}(\lambda_1)=\mathrm{Re}(\lambda_2)=\mathrm{tr}(J(x_*,y_*))>0,$$

于是

$$\mathrm{Im}(\lambda_1)>\mathrm{Re}(\lambda_1)\tan\left(\frac{\alpha\pi}{2}\right),$$

$$-\mathrm{Im}(\lambda_2)>\mathrm{Re}(\lambda_2)\tan\left(\frac{\alpha\pi}{2}\right),$$

这意味着

$$\frac{\alpha\pi}{2}<\arg(\lambda_1)<\frac{\pi}{2},$$

$$-\frac{\pi}{2}<\arg(\lambda_2)<-\frac{\alpha\pi}{2},$$

故

$$|\arg(\lambda_1)|>\frac{\alpha\pi}{2},|\arg(\lambda_2)|>\frac{\alpha\pi}{2}.$$

由引理 1 可知，在条件(ii)下正平衡点 $E_3(x_*,y_*)$ 局部渐近稳定.　】

定理 6　若 $e\sigma<\delta,\beta(c-e)<rde,\mathrm{tr}(J(x_*,y_*))>0$，且下列条件之一成立：

(i) $\mathrm{tr}^2(J(x_*,y_*))-4\det(J(x_*,y_*))\geqslant 0$;

(ii) $\mathrm{tr}^2(J(x_*,y_*))-4\det(J(x_*,y_*))<0$,

$$|\mathrm{tr}^2(J(x_*,y_*))-4\det(J(x_*,y_*))|^{\frac{1}{2}}<\mathrm{tr}(J(x_*,y_*))\tan\frac{\alpha\pi}{2}.$$

则系统(1)的正平衡点 $E_3(x_*,y_*)$ 不稳定.

证明　若条件(i)成立，则其特征方程至少有一个正根，此即意味着平衡点 $E_3(x_*,y_*)$ 不稳定.

若条件(ii)成立，则 λ_1 和 λ_2 是一对共轭复根，且

$$\mathrm{Im}(\lambda_1)=-\mathrm{Im}(\lambda_2)>0,$$

$$\mathrm{Re}(\lambda_1)=-\mathrm{Re}(\lambda_2)>0.$$

结合条件(2)，有

$$\mathrm{Im}(\lambda_1)<\mathrm{Re}(\lambda_1)\tan\left(\frac{\alpha\pi}{2}\right),$$

$$-\mathrm{Im}(\lambda_2)<\mathrm{Re}(\lambda_2)\tan\left(\frac{\alpha\pi}{2}\right),$$

即

$$0<\arg(\lambda_1)<\frac{\alpha\pi}{2},\ -\frac{\alpha\pi}{2}<\arg(\lambda_2)<0.$$

故

$$|\arg(\lambda_1)|<\frac{\alpha\pi}{2},\ |\arg(\lambda_2)|<\frac{\alpha\pi}{2},$$

结合引理 1 知平衡点 $E_3(x_*,y_*)$ 不稳定.　】

2.4　平衡点的全局渐近稳定性

定理 7　若 $e\sigma<\delta,\beta(c-e)<rde$，且 $\frac{r}{K}>\frac{\beta\sigma y_*}{(1+\sigma x_*)}$，则系统(1)的正平衡点 $E_3(x_*,y_*)$ 是

全局渐近吸引的.

证明　定义 $E_3(x_*,y_*)$ 处的 Lyapunov 函数 $V(t)$ 为

$$V(t)=\left(x(t)-x_*-x_*\ln\frac{x}{x_*}\right)+L\left(y(t)-y_*-y_*\ln\frac{y}{y_*}\right),$$

其中 $L>0$，则由引理2可得

$$\begin{aligned}\mathrm{D}^\alpha V(t)\leqslant&\frac{x(t)-x_*}{x(t)}\mathrm{D}^\alpha x(t)+L\frac{y(t)-y_*}{y(t)}\mathrm{D}^\alpha y(t)\leqslant\\&-\left(\frac{r}{K}-\frac{\beta\sigma y_*}{(1+\sigma x)(1+\sigma x_*)}\right)(x-x_*)^2-\frac{\beta(x-x_*)(y-y_*)}{1+\sigma x}+\\&\frac{L\delta(x-x_*)(y-y_*)}{(1+\sigma x)(1+\sigma x_*)}-\frac{Lcd(y-y_*)^2}{(1+dy)(1-dy_*)}.\end{aligned}$$

取 $L=\dfrac{\beta}{\delta}(1+\sigma x_*)$，由假设

$$\frac{r}{K}>\frac{\beta\sigma y_*}{(1+\sigma x_*)}>\frac{\beta\sigma y_*}{(1+\sigma x)(1+\sigma x_*)}$$

即得

$$\begin{aligned}\mathrm{D}^\alpha V(t)\leqslant&-\left(\frac{r}{K}-\frac{\beta\sigma y_*}{(1+\sigma x)(1+\sigma x_*)}\right)(x-x_*)^2-\\&\frac{Lcd(y-y_*)^2}{(1+dy)(1+dy_*)}\leqslant 0,\end{aligned}$$

于是 $\mathrm{D}^\alpha V(t)\leqslant 0$，且当 $(x,y)\neq(x_*,y_*)$ 时 $\mathrm{D}^\alpha V(t)<0$. 于是对任意初值 $(x(t_0),y(t_0))\in\Omega_+$，系统(1)的解有界. 此外，零平衡点 $E_0(0,0)$，边界平衡点 $E_1(K,0)$ 和 $E_2\left(0,\dfrac{c-e}{ed}\right)$ 均不稳定. 因为集合 $\bar{\Gamma}:=\{(x,y):\mathrm{D}^\alpha V(t)=0\}$ 只包含 $\{E_3\}$，根据文献[16]的引理4.6知，对任意初值 $(x(t_0),y(t_0))\in\Omega_+$，当 $t\to+\infty$ 时，$(x(t),y(t))\to(x_*,y_*)$，即正平衡点 $E_3(x_*,y_*)$ 是全局吸引的，因而系统(1)的正平衡点 $E_3(x_*,y_*)$ 是全局渐近稳定的.　】

3　数值模拟

例1　取 $r=0.6,K=1,\sigma=40,\beta=0.04,\delta=20,c=0.5,d=1,d=0.48,\alpha=0.99$.显然不等式

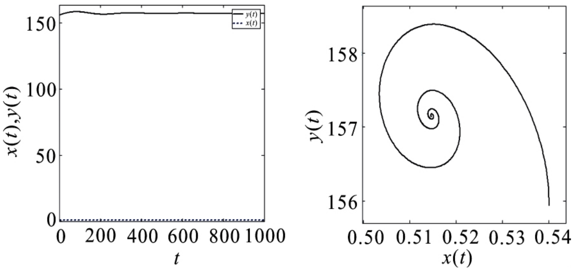


图1　平衡点 $E_3(0.5146,157.15)$ 局部渐近稳定

Fig 1　Local asymptotic stability of equilibrium $E_3(0.5146,157.15)$

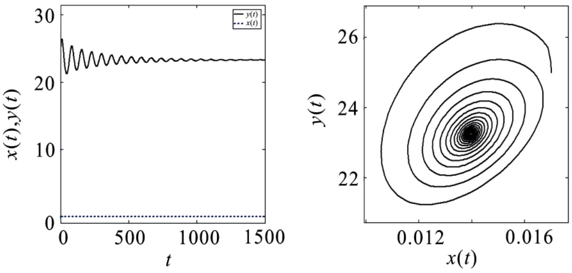
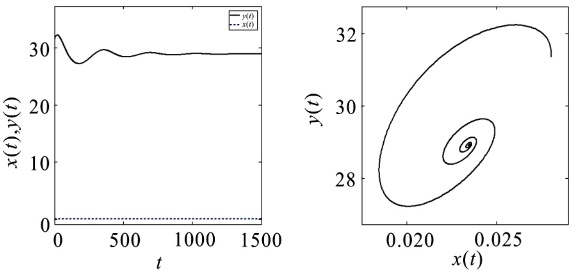
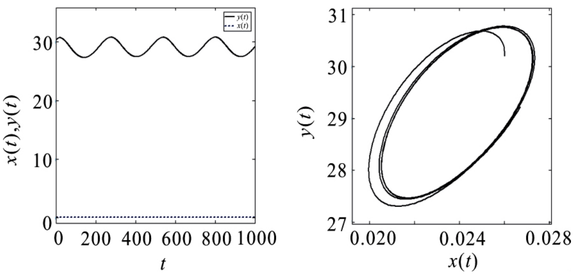


图2　平衡点 $E_3(0.0139848,23.2273)$ 局部渐近稳定

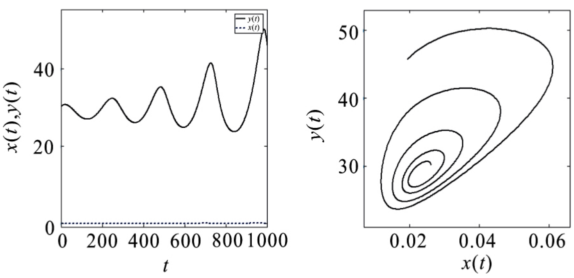
Fig 2　Local asymptotic stability of equilibrium $E_3(0.0139848,23.2273)$



(a) $\alpha=0.53$



(b) $\alpha=0.58$



(c) $\alpha=0.612$

图3　系统稳定性的动态变化过程

Fig 3　Dynamic evolution of stability for the system

$e\sigma < \delta$，$\beta(c-e) < rde$ 成立，且条件 $\mathrm{tr}(J(x_*, y_*)) = -0.034 \leqslant 0$ 亦满足，所以由定理 5 的条件（i）知平衡点 $E_3(x_*, y_*) = (0.5146, 157.15)$ 局部渐近稳定（图 1）.

若取 $K=2$，$e=0.2$，$\alpha=0.71$，其他参数不变，则不等式 $e\sigma < \delta$ 和 $\beta(c-e) < rde$ 均成立，且定理 5 的条件（ii）亦成立，所以平衡点 $E_3(x_*, y_*) = (0.0139848, 23.2273)$ 局部渐近稳定（图 2）.

取 $K=2$，$\delta=15$，其他参数不变，α 分别取 0.53，0.58 和 0.612，两不等式仍然成立，图 3(a)～(c) 体现了相应的平衡点从稳定到不稳定的变化过程.

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一类具有脉冲捕食食饵系统的动力学分析

蒲武军
(陇南师范高等专科学校 数学与信息科学学院,甘肃 陇南 742500)

摘要:讨论一类具有脉冲效应的反应扩散三种群捕食系统在齐次 Neumann 边界条件下的动力学行为,利用比较方法,得到了该系统的持久性,以及周期解存在和渐近稳定的充分条件.
关键词:捕食系统;持久性;周期解;反应扩散;脉冲
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Dynamic analysis of a predator-prey system with impulse

PU Wu-jun
(College of Mathematics and Information Science, Longnan Teachers College, Longnan 742500, Gansu, China)

Abstract: A impulsive three-species reaction-diffusion predator-prey system with homogeneous Neumann boundary condition is investigated. Based on the comparison arguments, the permanence of the system is established, and the existence and asymptotic stability of periodic solution of the system are also given.
Key words: predator-prey system; permanence; periodic solution; reaction-diffusion; impulsive

在种群动力学中,时间和空间的非均匀性对物种持续生存起着重要作用,因此研究具有扩散的数学模型具有实际意义.通常,在生态环境中,脉冲和扩散现象经常发生,如某些动物的季节性繁殖,渔业养殖与森林管理中的投放、收获等都是脉冲现象[1].同时,由于种群密度分布是不均匀的,高密度位置的种群会向低密度位置扩散,所以研究受脉冲影响的扩散系统显得十分必要.近年来,运用反应扩散方程理论研究生态学领域中的捕食系统已经受到了学者们的广泛关注[2-8].

受文献[2]的启发,文中讨论如下 3 种群非自治捕食系统:

$$\frac{\partial u_1}{\partial t}=d_1\Delta u_1+u_1\left(1-u_1-\frac{a_1(t,x)u_2}{u_1+u_2}\right),\quad (t,x)\in\Sigma,\ t\neq t_k,\tag{1}$$

$$\frac{\partial u_2}{\partial t}=d_2\Delta u_2+u_2\left(\frac{m_1(t,x)u_1}{u_1+u_2}-b_1(t,x)\frac{a_2(t,x)u_3}{u_2+u_3}\right),\quad (t,x)\in\Sigma,\ t\neq t_k,\tag{2}$$

$$\frac{\partial u_3}{\partial t}=d_3\Delta u_3+u_3\left(\frac{m_2(t,x)u_2}{u_2+u_3}-b_2(t,x)\right),\quad (t,x)\in\Sigma,\ t\neq t_k,\tag{3}$$

$$\left.\frac{\partial u_i}{\partial n}\right|_{\partial\Omega}=0,\ i=1,2,3,\quad (t,x)\in\bar{\Sigma},t>0,\tag{4}$$

$$u_i(t_k+0,x)=u_i(t_k,x)f_{ik}(x,u_1,u_2,u_3),\quad i=1,2,3,\ k=1,2,\cdots\tag{5}$$

其中,$\Omega\subset\mathrm{R}^n$ 是一个具有光滑边界 $\partial\Omega$ 的有界区

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作者简介:蒲武军(1979—),男,甘肃庄浪人,讲师,硕士.主要研究方向为应用微分方程和生物数学.
E-mail: puwj2005@163.com

域；n 是定义在 $\partial\Omega$ 上的单位外法向量，齐次 Neumann 边界条件表明系统是自封的，即在边界 $\partial\Omega$ 上食物和捕食者的进出流量为零；$d_i>0(i=1,2,3)$表示物种的扩散系数；$\Delta u=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}$是拉普拉斯算子．有关系统（1）～（5）的生物学意义参见文献[3]．

文中主要讨论系统（1）～（5）的持久性以及周期解的存在性和渐近稳定性．

1　预备知识

始终假设：

(H_1) $a_1(t,x),a_2(t,x),m_1(t,x),m_2(t,x),b_1(t,x),b_2(t,x)$均是 $\mathsf{R}\times\bar{\Omega}$ 上的有界正值函数，且关于 t,x 连续可微，关于 t 是 ω 周期的；

(H_2) $f_{ik}(x,u_1,u_2,u_3),i=1,2,3$ 是正值函数，且关于各变元连续可微；

(H_3) 实数序列$\{t_k\}$满足 $0=t_0<t_1<t_2<\cdots<t_k<\cdots$，且存在正整数 $p\in\mathsf{N}$，使得 $t_{k+p}=t_k+\omega,k\geqslant 1$；

(H_4) 对 x,u_1,u_2,u_3 和 $k\geqslant 1$，序列 $f_{ik}(x,u_1,u_2,u_3),i=1,2,3$ 满足下列等式：$f_{ik+p}(x,u_1,u_2,u_3)=f_{ik}(x,u_1,u_2,u_3),i=1,2,3$.

为方便起见，引入以下记号：

$$\mathsf{R}_+=[0,\infty),\ G=\mathsf{R}_+\times\Omega,\ \bar{G}=\mathsf{R}_+\times\bar{\Omega},$$

$$\Sigma_k=\{(t,x),t\in(t_{k-1},t_k),x\in\Omega,k\in\mathsf{N}\},$$

$$\Sigma=\bigcup_{k\in\mathsf{N}}\Sigma_k,\ \bar{\Sigma}_k=\{(t,x),t\in(t_{k-1},t_k),x\in\bar{\Omega}\},\ k\in\mathsf{N},\ \bar{\Sigma}=\bigcup_{k\in\mathsf{N}}\bar{\Sigma}_k.$$

记 $\mathcal{R}$ 为所有函数 $\varphi:\bar{G}\longrightarrow\mathsf{R}$ 构成的集合，其中 φ 满足 $\varphi(t,x)\in C^{1,2}_{t,x}(\Sigma_k),\varphi(t,x)\in C^{1,1}_{t,x}(\bar{\Sigma}_k)$，且对 $\forall x\in\Omega,k\in\mathsf{N}$，存在极限 $\lim\limits_{s\to t_k-0}\varphi(s,x)=\varphi(t_k,x)$，$\lim\limits_{s\to t_k+0}\varphi(s,x)=\varphi(t_k+0,x)$．称$(u_1(t,x),u_2(t,x),u_3(t,x))\in\mathcal{R}\times\mathcal{R}\times\mathcal{R}$ 为系统（1）～（5）的解，若$(u_1(t,x),u_2(t,x),u_3(t,x))$在 Σ 上满足方程（1）～（3），且当 $x\in\partial\Omega$ 时满足（4）和（5）式．

对有界函数 $\varphi(t,x)$，记

$$\varphi^{\mathrm{L}}=\inf_{(t,x)}\varphi(t,x),\ \varphi^{\mathrm{M}}=\sup_{(t,x)}\varphi(t,x).$$

为证明系统（1）～（5）的持久性，先考虑如下的脉冲 Logistic 方程：

$$\begin{cases}z'(t)=az(b-z),\ t\neq t_k,\\ z(t_k+0)=z(t_k)\lambda_k(t_k),\ k\in\mathsf{N},\end{cases}\tag{6}$$

其中 $z\in\mathsf{R}_+,t_k$ 满足条件(H_3)，λ_k 为连续正值函数，且 $\lambda_{k+p}(z)=\lambda_k(z)$.

引理 1[2]　系统（6）的每个解 $z(t)=z(t,0,t_0),z_0>0$ 满足 $0<z(t)\leqslant C$，其中 $C>0$，当 $t\geqslant\theta$ 时，$\theta=\min\limits_{i\in\{1,2,\cdots,p\}}(t_{i+1}-t_i)$.

引理 2[9]　设向量值函数

$$v(t,x)=(v_1(t,x),v_2(t,x),\cdots,v_m(t,x)),$$

$$w(t,x)=(w_1(t,x),w_2(t,x),\cdots,w_m(t,x))$$

满足：

（i）关于 $x\in\Omega$ 是 C^2 的，关于$(t,x)\in[a,b]\times\bar{\Omega}$ 是 C^1 的，其中 Ω 是 R^n 中边界光滑的有界区域；

（ii）对于$(t,x)\in[a,b]\times\Omega,\mu=(\mu_1,\mu_2,\cdots,\mu_m)>0$（分量意义下），有

$$v_t-\mu\Delta v-g(t,x,v)\leqslant w_t-\mu\Delta w-g(t,x,w),$$

其中 $g(t,x,u)=(g_1(t,x,u),\cdots,g_m(t,x,u))$关于 $u=(u_1,u_2,\cdots,u_m)$连续可微，且拟单调增加，即

$$\frac{\partial g_i(t,x,u_1,u_2,\cdots,u_m)}{\partial g_j}\geqslant 0,$$

$$i,j=1,\cdots,m,\ i\neq j;$$

（iii）$\frac{\partial v}{\partial n}=\frac{\partial w}{\partial n}$，$(t,x)\in[a,b]\times\partial\Omega$.

则 $v(t,x)\leqslant w(t,x),(t,x)\in[a,b]\times\bar{\Omega}$.

引理 3[10]　设向量值函数 $u(t,x)$在$[0,T]\times\bar{\Omega}$ 上连续，在$(0,T)\times\Omega$ 上连续可微，且 $u(t,x)$满足

$$\begin{cases}u_t-d\Delta u+c(t,x)u\geqslant 0,\ (t,x)\in(0,T]\times\Omega,\\ \frac{\partial u}{\partial n}=0,\ (t,x)\in(0,T]\times\partial\Omega,\\ u(0,x)\geqslant 0,\ x\in\Omega,\end{cases}$$

其中 $c(t,x)$在$(0,T]\times\Omega$ 上有界，则 $u(t,x)\geqslant 0$，$(t,x)\in(0,T]\times\bar{\Omega}$．进一步，若 $u(0,x)$不恒为零，则 $u(t,x)>0,(t,x)\in(0,T]\times\bar{\Omega}$.

为证明系统（1）～（5）周期解的存在性，还需引入以下引理．为了便于叙述，首先将系统（1）～（5）改写为以下形式：

$$\frac{\mathrm{d}w}{\mathrm{d}t}=A_1w+F(t,w),\ t\neq t_k,\tag{7}$$

$$w(t_i+0)=w(t_i)+G_i(w(t_i)),\ i\in\mathsf{N},\tag{8}$$

其中 $w=(u_1,u_2,u_3)\in L_p\times L_p\times L_p,p(p>n)$是正

整数，

$$A_1=\begin{pmatrix} d_1\Delta-\delta & 0 & 0\\ 0 & d_2\Delta-\delta & 0\\ 0 & 0 & d_3\Delta-\delta\end{pmatrix},\ \delta>0,$$

$$F(t,w)=$$

$$\begin{pmatrix} u_1\left(1-u_1-\dfrac{a_1(t,x)u_2}{u_1+u_2}\right)+\delta u_1\\ u_2\left(\dfrac{m_1(t,x)u_1}{u_1+u_2}-b_1(t,x)-\dfrac{a_2(t,x)u_3}{u_2+u_3}\right)+\delta u_2\\ u_3\left(\dfrac{m_2(t,x)u_2}{u_2+u_3}-b_2(t,x)\right)+\delta u_3\end{pmatrix},$$

$$G_i(w(t_i))=\begin{pmatrix} u_1(t_i,x)f_{1i}(x,u_1(t_i,x),\\ \quad u_2(t_i,x),u_3(t_i,x))-u_1(t_i,x)\\ u_2(t_i,x)f_{2i}(x,u_1(t_i,x),\\ \quad u_2(t_i,x),u_3(t_i,x))-u_2(t_i,x)\\ u_3(t_i,x)f_{3i}(x,u_1(t_i,x),\\ \quad u_2(t_i,x),u_3(t_i,x))-u_3(t_i,x)\end{pmatrix}.$$

记算子 A_1 的定义域为

$$D(A_1)=\left\{\zeta:\zeta\in W^{2,p}(\Omega),\frac{\partial\zeta}{\partial n}\bigg|_{\partial\Omega}=0\right\},$$

其中 $W^{2,p}(\Omega)$ 是 L_p 上具有两阶广义导数的 Sobolev 空间. 由文献[11]可知 A_1 是扇形算子.

引理 4[2] 设函数 G_i 连续可微，并且存在一个正值函数 $\gamma(M)$ 使得

$$\sup_{\|w\|_\alpha\leqslant M}\|G_k(w)\|_\alpha\leqslant\gamma(M),$$

$$k\in\mathrm{N},\alpha\in\left(\frac{1}{2}+\frac{n}{2p},1\right),\qquad(9)$$

其中 $w(t,w_0),w_0=(u_{10},u_{20},u_{30})\in X^\alpha$ 是方程(15)和(16)的有界解，即 $\|w(t,w_0)\|_C\leqslant N,t>0$. 则集合 $\{w(t,w_0):t>0\}$ 是 $C^{1+v}(\bar\Omega,\mathrm{R}^2)$ 中的相对紧集，其中 $0<v<2\alpha-1-n/p$.

2 主要结果

本节证明系统(1)～(5)的持久性和周期解的存在性.

定理 1 设条件(H_1)～(H_4)满足，则 R^3_+ 为系统(1)～(5)的正向不变集.

证明 设

$$(u_1(t,x,u_{10},u_{20},u_{30}),u_2(t,x,u_{10},u_{20},u_{30}),\ u_3(t,x,u_{10},u_{20},u_{30}))$$

是系统(1)～(5)满足非负且不恒为零的初值 $u_{10}(x),u_{20}(x),u_{30}(x)$ 的解，$\hat u_1(t,x),\tilde u_1(t,x)$ 分别满足初值问题：

$$\begin{cases}\dfrac{\partial\hat u_1}{\partial t}-d_1\Delta\hat u_1-\hat u_1(1-a_1^{\mathrm M}-\hat u_1)=0,\\ \hat u_1(0,x)=u_{10}(x),\end{cases}$$

$$\begin{cases}\dfrac{\partial\tilde u_1}{\partial t}-d_1\Delta\tilde u_1-\tilde u_1(1-\tilde u_1)=0,\\ \tilde u_1(0,x)=u_{10}(x).\end{cases}$$

则容易验证 $\hat u_1(t,x),\tilde u_1(t,x)$ 分别是方程(1)的下解和上解. 考虑到 $u_{10}(x)$ 非负且不恒为零，根据引理 3，有 $\hat u_1(t,x)>0,\tilde u_1(t,x)>0,t\in(0,t_1]$. 由比较原理知 $u_1(t,x)>0,t\in(0,t_1]$. 注意到 f_1 是正的，同理可证 $u_1(t,x)>0,t\in[t_1,t_2]$. 由归纳法可知 $u_1(t,x)>0,t\in(0,\infty)$.

对方程(2)和(3)，同理可证

$$u_2(t,x)>0,u_3(t,x)>0,\ t\in(0,\infty).\qquad】$$

定理 2 设条件(H_1)～(H_4)满足，进一步假设下面条件成立：

(H_5) 存在函数 $\eta(M)>0$，当 $u_1\leqslant M,u_2\geqslant0,x\in\Omega$ 时，$f_{1k}(x,u_1,u_2,u_3)\leqslant\eta(M),k\in\mathrm{N}$.

(H_6) $-\omega b_1^{\mathrm L}+\sum\limits_{i=1}^{p}\ln f_{2i}<0$，其中

$$f_{2i}=\sup_{(x,u_1,u_2,u_3)}f_{2i}(x,u_1,u_2,u_3).$$

(H_7) $-\omega b_2^{\mathrm L}+\sum\limits_{i=1}^{p}\ln f_{3i}<0$，其中

$$f_{3i}=\sup_{(x,u_1,u_2,u_3)}f_{3i}(x,u_1,u_2,u_3).$$

则系统(1)～(5)满足非负且不恒等于零的初值条件的解最终有界，即存在 $\bar t=\bar t(u_{10},u_{20},u_{30})>0$ 和常数 $N_i>0(i=1,2,3)$，使得当 $t>\bar t$ 时系统(1)～(5)的解 $(u_1(t,x,u_{10},u_{20},u_{30}),u_2(t,x,u_{10},u_{20},u_{30}),u_3(t,x,u_{10},u_{20},u_{30}))$ 满足 $u_i(t,x,u_{10},u_{20},u_{30})\leqslant N_i,i=1,2,3,x\in\bar\Omega$.

证明 设 $\bar u_1(t,x,u_{10})$ 是方程

$$\frac{\partial\bar u_1}{\partial t}-d_1\Delta u_1-u_1(1-\bar u)=0$$

满足初值为 u_{10} 的解. 由于

$$0=\frac{\partial u_1}{\partial t}-d_1\Delta u_1-u_1\left(1-u_1-\frac{a_1(t,x)u_2}{u_1+u_2}\right)\geqslant$$

$$\frac{\partial\bar u_1}{\partial t}-d_1\Delta\bar u_1-\bar u_1(1-\bar u_1),$$

$$0=\frac{\partial\bar u_1}{\partial t}-d_1\Delta\bar u_1-\bar u_1(1-\bar u_1)\geqslant$$

$$\frac{\partial u_1}{\partial t}-d_1\Delta u_1-u_1(1-u_1),$$

则由引理 3 知 $u_1(t,x,u_{10},u_{20},u_{30})\leqslant\bar u_1(t,M_{u_1})$,

其中 M_{u_1} 满足 $\|u_{10}(x)\|_C=\max\limits_{x\in\Omega}|u_{10}(x)|\leqslant M_{u_1}$. 注意到 $\bar{u}_1(t,M_{u_1})$ 独立于 x, 故 $\bar{u}_1(t,M_{u_1})$ 为初值问题

$$\begin{cases}\dfrac{\mathrm{d}\bar{u}_1}{\mathrm{d}t}=\bar{u}_1(1-\bar{u}_1),\\ \bar{u}_1(0,M_{u_1})=M_{u_1}\end{cases}$$

的解. 由条件(H_5)可知

$$\|u_1(t_k+0,x,u_{10},u_{20},u_{30})\|_C\leqslant \bar{u}_1(t_k,M_{u_1})\eta(\bar{u}_1(t_k,M_{u_1})).$$

根据引理1可知脉冲微分方程

$$\begin{cases}\dfrac{\mathrm{d}\bar{u}_1}{\mathrm{d}t}=\bar{u}_1(1-\bar{u}_1),\\ \bar{u}_1(t_k+0)=\eta(\bar{u}_1(t_k))\end{cases}$$

的解是最终有界的, 进而方程(1)和(5)的解最终有界, 即存在常数 $N_1>0$, 使得

$$u_1(t,x)\leqslant N_1,\ \forall x\in\bar{\Omega},t>\bar{t}.$$

根据方程(2), 有

$$0=\frac{\partial u_2}{\partial t}-d_2\Delta u_2-u_2\left(\frac{m_1(t,x)u_1}{u_1+u_2}-b_1(t,x)-\frac{a_2(t,x)u_3}{u_2+u_3}\right)\geqslant \frac{\partial u_2}{\partial t}-d_2\Delta u_2+b_1^{\mathrm{L}}u_2-m_1^{\mathrm{M}}N_1.$$

设 $\bar{u}_2(t,M_{u_2})$ 为初值问题

$$\begin{cases}\dfrac{\mathrm{d}\bar{u}_2}{\mathrm{d}t}=-b_1^{\mathrm{L}}\bar{u}_2+m_1^{\mathrm{M}}N_1,\\ \bar{u}_2(0,M_{u_2})=M_{u_2}\end{cases}$$

的解, 其中常数 M_{u_2} 满足

$$\|u_{20}(x)\|_C=\max_{x\in\Omega}|u_{20}(x)|\leqslant M_{u_2},$$

则由引理3可知

$$u_2(t,x,u_{10},u_{20},u_{30})\leqslant\bar{u}_2(t,M_{u_2}).$$

由文献[10]知, 线性周期脉冲微分方程

$$\begin{cases}\dfrac{\mathrm{d}\bar{u}_2}{\mathrm{d}t}=-b_1^{\mathrm{L}}\bar{u}_2+m_1^{\mathrm{M}}N_1,\\ \bar{u}_2(t_k+0)=f_{2k}\bar{u}_2(t_k)\end{cases}\tag{10}$$

具有形如 $\bar{u}_2(t)=X_0(t)+AX(t)$ 的解, 其中常数 $A>0$, $X_0(t)$ 是分段连续的 ω 周期函数, 并且 $X(t)=\exp\left(-b_1^{\mathrm{L}}t+\sum\limits_{0<t_k<t}\ln f_{2k}\right)$. 由条件($H_6$)可知, 当 $t\to\infty$ 时 $X(t)\to 0$. 故而方程(10)的解也最终有界, 因此(2)和(5)的解也是最终有界的.

方程(3)的证明完全类似于方程(2), 只需结合条件(H_7)即可证明, 在此不再赘述. 】

定理3 设条件(H_1)~(H_4)满足, 进一步假设系统(1)~(5)满足非负初值的解是最终有界的, 且下列不等式成立:

$$\sum_{i=1}^{p}\ln\inf_{x\in\Omega,(u_1,u_2,u_3)\in S}f_{1i}(x,u_1,u_2,u_3)+\omega(1-a_1^{\mathrm{M}})>0,\tag{11}$$

$$\sum_{i=1}^{p}\ln\inf_{x\in\Omega,(u_1,u_2,u_3)\in S}f_{2i}(x,u_1,u_2,u_3)+\omega(m_1^{\mathrm{L}}-b_1^{\mathrm{M}}-a_2^{\mathrm{M}})>0,\tag{12}$$

$$\sum_{i=1}^{p}\ln\inf_{x\in\Omega,(u_1,u_2,u_3)\in S}f_{3i}(x,u_1,u_2,u_3)+\omega(m_2^{\mathrm{L}}-b_2^{\mathrm{M}})>0,\tag{13}$$

其中 $S=\{(u_1,u_2,u_3):0<u_j\leqslant N_j\}$, $N_j(j=1,2,3)$ 为定理2中所定义. 则系统(1)~(5)是持久的, 即存在常数 $\xi_j>0,j=1,2,3$, 对满足不恒为零的非负初值 $u_{10}(x),u_{20}(x),u_{30}(x)$ 的解 u_j, 存在 $\tilde{t}=\tilde{t}(u_{10},u_{20},u_{30})$, 当 $t>\tilde{t}$ 时, 有

$$\xi_j<u_j\leqslant N_j,\ j=1,2,3,\forall x\in\bar{\Omega}.$$

证明 引理3表明, 若 $u_{10}(x),u_{20}(x),u_{30}(x)$ 非负且不恒为零, 所以

$$u_i(t,x,u_{10},u_{20},u_{30})>0,\ i=1,2,3,x\in\bar{\Omega},t>0.$$

不失一般性, 设 $\min\limits_{x\in\Omega}u_{i0}(x)=l_{u_i},i=1,2,3$. 由

$$\frac{\partial u_1}{\partial t}-d_1\Delta u_1-u_1\left(1-u_1-\frac{a_1(t,x)u_2}{u_1+u_2}\right)\leqslant \frac{\partial u_1}{\partial t}-d_1\Delta u_1-u_1(1-a_1^{\mathrm{M}}-u_1)$$

可得

$$0=\frac{\partial\hat{u}_1}{\partial t}-d_1\Delta\hat{u}_1-\hat{u}_1(1-a_1^{\mathrm{M}}-\hat{u}_1)\leqslant \frac{\partial u_1}{\partial t}-d_1\Delta u_1-u_1(1-a_1^{\mathrm{M}}-u_1).$$

由引理2可知, 当 $m=1$ 时,

$$u_1(t,x,u_{10},u_{20},u_{30})\geqslant\hat{u}_1(t,l_{u_1}),\ t\in[0,t_1].$$

于是当 $t=t_1$ 时, 结合方程(5)有

$$u_1(t_1+0,x,u_{10},u_{20},u_{30})\geqslant \hat{u}_1(t_1,M_{u_1})\inf_{x\in\Omega,(u_1,u_2,u_3)\in S}f_{11}(x,u_1,u_2,u_3).$$

结合周期脉冲 Logistic 方程

$$\begin{cases}\dfrac{\mathrm{d}\hat{u}_1}{\mathrm{d}t}=\hat{u}_1(1-a_1^{\mathrm{M}}-\hat{u}_1),\\ \hat{u}_1(t_i+0)=\hat{u}_1(t_i)\inf\limits_{x\in\Omega,(u_1,u_2,u_3)\in S}f_{1i}(x,u_1,u_2,u_3),\end{cases}\tag{14}$$

可知 $u_1(t,x,u_{10},u_{20},u_{30})$ 有界. 由文献[12]中定理2.1及条件(11)可知, 方程(14)有唯一分段连续且严格正的周期解 $\hat{u}_1^*(t)$, 使得对于满足 $\hat{u}_{1l}>0$ 的解 $\hat{u}_1(t,u_{1l})$ 有

$$\hat{u}_1(t,u_{1l}) \to \hat{u}_1^*(t),\ t \to \infty.$$

因此，存在常数 $\xi_1>0,\bar{t}=\bar{t}(t_{1l})>0$，使得当 $t>\bar{t}$ 时，$u_1(t,x,u_{10},u_{20},u_{30})\geqslant\xi_1$.

由于 $u_1(t,x,u_{10},u_{20},u_{30})\geqslant\xi_1$，所以

$$0=\frac{\partial u_2}{\partial t}-d_2\Delta u_2-u_2\left(\frac{m_1(t,x)u_1}{u_1+u_2}-b_1(t,x)-\frac{a_2(t,x)u_3}{u_2+u_3}\right)\leqslant\frac{\partial u_2}{\partial t}-d_2\Delta u_2+(b_1^{\mathrm{M}}+a_2^{\mathrm{M}}-m_1^{\mathrm{L}})u_2+\frac{m_1^{\mathrm{L}}u_2^2}{\xi_1}.$$

故 $u_2(t,x,u_{10},u_{20},u_{30})\geqslant\hat{u}_2(t,l_{u_2})$，其中 $\hat{u}_2(t,l_{u_2}),\hat{u}_2(0,l_{u_2})=l_{u_2}$ 是方程

$$\begin{cases}\dfrac{\mathrm{d}\hat{u}_2}{\mathrm{d}t}=(m_1^{\mathrm{L}}-b_1^{\mathrm{M}}-a_2^{\mathrm{M}})\hat{u}_2-\dfrac{m_1^{\mathrm{L}}\hat{u}_2^2}{\xi_1},\\ \hat{u}_2(t_i+0)=\hat{u}_2(t_i)\hat{f}_{2i}\end{cases}\tag{15}$$

的解，其中 $\hat{f}_{2i}=\inf\limits_{x\in\Omega,(u_1,u_2,u_3)\in S}f_{2i}(x,u_1,u_2,u_3)$. 若 $\hat{u}_2(t)\leqslant\xi_2,t\in[0,t_1]$，则有

$$\hat{u}_2(t,l_{u_2})\geqslant l_{u_2}\exp\left\{t_1\left(m_1^{\mathrm{L}}-b_1^{\mathrm{M}}-a_2^{\mathrm{M}}-\frac{m_1^{\mathrm{L}}\xi_2}{\xi_1}\right)\right\},$$

$$\hat{u}_2(t_1+0,l_{u_2})\geqslant\hat{f}_{21}l_{u_2}\exp\left\{t_1\left(m_1^{\mathrm{L}}-b_1^{\mathrm{M}}-a_2^{\mathrm{M}}-\frac{m_1^{\mathrm{L}}\xi_2}{\xi_1}\right)\right\}.$$

因此，若 $\hat{u}_2(t)\leqslant\xi_2,t\in[0,\omega]$，则

$$\hat{u}_2(\omega,l_{u_2})\geqslant l_{u_2}\exp\left\{\sum_{i=1}^{p}\ln\hat{f}_{2i}+\omega\left(m_1^{\mathrm{L}}-b_1^{\mathrm{M}}-a_2^{\mathrm{M}}-\frac{m_1^{\mathrm{L}}\xi_2}{\xi_1}\right)\right\}.$$

结合(12)式，可取充分小的 $\xi_2^*>0$，使得

$$\sum_{i=1}^{p}\ln\hat{f}_{2i}+\omega\left(m_1^{\mathrm{L}}-b_1^{\mathrm{M}}-a_2^{\mathrm{M}}-\frac{m_1^{\mathrm{L}}\xi_2}{\xi_1}\right)=\rho>0.$$

取 $\xi_2^{**}\in(0,\xi_2^*)$，则存在正整数 k_2，使得

$$\hat{u}_2(k_2\omega,\hat{u}_{20})\geqslant\mathrm{e}^{k_2\rho}l_{u_2}\geqslant\xi_2^{**}(\hat{u}_2(t,l_{u_2}))<\xi_2^*,\ t\in[0,k_2\omega].$$

因此，对于方程(15)的每个解 $\hat{u}_2(t,\hat{u}_{20}),\hat{u}_{20}>0$，存在 $\hat{t}$，使得当 $t>\hat{t}$ 时，$\hat{u}_2(t,\hat{u}_{20})\geqslant\xi_2^{**}$.

设 $\hat{u}_2(t,\tau,\hat{u}_{20})$ 是方程(15)满足初值条件 $\hat{u}_2(\tau,\tau,\hat{u}_{20})=\hat{u}_{20}$ 的解，令

$$\xi_2=\inf\{\hat{u}_2(t,\tau,\hat{u}_{20}):\tau\in[0,\omega],\ \hat{u}_{20}\in[\xi_2^{**},N_2],t\in[\tau,2\omega]\}.$$

则 $\hat{u}_2(t,\tau,\hat{u}_{20})\geqslant\xi_2,t\geqslant2\omega$. 事实上，考虑解

$$\hat{u}_2(t,\omega,\hat{u}_{20}),\ \hat{u}_{20}\geqslant\xi_\omega,$$

其中

$$\xi_w=\inf\{\hat{u}_2(\omega,\tau,\hat{u}_{20}):\tau\in[0,\omega],\ \hat{u}_{20}\in[\xi_2^{**},N_2]\}\geqslant\xi_2.$$

若 $\hat{u}_2(t,\omega,\hat{u}_{20})\leqslant\xi_2^*,t\in[\tau,2\omega]$，则

$$\hat{u}_2(2\omega,\omega,\hat{u}_{20})\geqslant\rho\hat{u}_2(\omega,\omega,\hat{u}_{20})\geqslant\xi_\omega.$$

若存在某个时刻 $t\in[\omega,2\omega]$，使得 $\hat{u}_2(t,\omega,\hat{u}_{20})>\xi_2^*$，则

$$\hat{u}_2(2\omega,\omega,\hat{u}_{20})\geqslant\xi_\omega.$$

因此，只需考虑 $\hat{u}_2(t,2\omega,\hat{u}_{20}),t\geqslant2\omega,\hat{u}_{20}\geqslant\xi_\omega$ 即可. 同理可证 $\xi_2,t\in[2\omega,3\omega]$ 之情形，如此继续即可完成证明.

方程(3)的证明完全类似于方程(2)，只需结合不等式(13)即可证明，不再赘述.　】

定理 4　假设条件$(\mathrm{H}_1)\sim(\mathrm{H}_4)$满足，且有

$$\sum_{i=1}^{p}\sup_{(x,u_1,u_2,u_3)}f_{2i}(x,u_1,u_2,u_3)+\omega(m_1^{\mathrm{M}}-b_1^{\mathrm{L}})<0,\tag{16}$$

$$\sum_{i=1}^{p}\sup_{(x,u_1,u_2,u_3)}f_{3i}(x,u_1,u_2,u_3)+\omega(m_2^{\mathrm{M}}-b_2^{\mathrm{L}})<0.\tag{17}$$

则当 $t\to\infty$ 时，$u_2(t,x)\to0,u_3(t,x)\to0$.

证明　取定 $M_{u_2}>0$，使得 $M_{u_2}\geqslant u_{20}(x)$. 设 $\bar{u}_2(t,M_{u_2})$ 是初值问题

$$\begin{cases}\dfrac{\mathrm{d}\bar{u}_2}{\mathrm{d}t}=(m_1^{\mathrm{M}}-b_1^{\mathrm{L}})\bar{u}_2,\\ \bar{u}_2(0,M_{u_2})=M_{u_2}\end{cases}$$

的解. 注意到

$$0=\frac{\partial u_2}{\partial t}-d_2\Delta u_2-u_2\left(\frac{m_1(t,x)u_1}{u_1+u_2}-b_1(t,x)-\frac{a_2(t,x)u_3}{u_2+u_3}\right)\geqslant\frac{\partial u_2}{\partial t}-d_2\Delta u_2+(b_1^{\mathrm{L}}-m_1^{\mathrm{M}})u_2,$$

根据比较原理知

$$u_2(t,x,u_{10},u_{20},u_{30})\leqslant\bar{u}_2(t,M_{u_2}),\ t\leqslant t_1.$$

根据脉冲条件(5)，有

$$u_2(t_1+0,x,u_{10},u_{20},u_{30})\leqslant\bar{u}_2(t,M_{u_2})\sup_{(x,u_1,u_2,u_3)}f_{21}(x,u_1,u_2,u_3).$$

结合脉冲线性方程

$$\begin{cases}\dfrac{\mathrm{d}\bar{u}_2}{\mathrm{d}t}=(m_1^{\mathrm{M}}-b_1^{\mathrm{L}})\bar{u}_2,\\ \bar{u}_2(t_k+0)=\bar{u}_2(t_k)\sup\limits_{(x,u_1,u_2,u_3)}f_{2k}(x,u_1,u_2,u_3)\end{cases}$$

知(2)和(5)的解有界. 进一步结合条件(16)知，

$$u_2(t,x)\to 0,\ t\to\infty.$$

$u_3(t,x)\to 0, t\to\infty$ 的证明和上述证明过程完全一致，在此不再加以证明. 】

定理5 假设(H_1)～(H_4)以及条件(17)满足，且系统(1)～(5)是持久的，即存在正常数 β 和 N，以及某个时刻 $\bar{t}=\bar{t}(u_{10},u_{20},u_{30})$，当 $t>\bar{t}$ 时，对于满足不恒为零的非负初值的任何解都有

$$(u_1,u_2,u_3)\in\Xi=\{(u_1,u_2,u_3):\beta\leqslant u_i(t,x)\leqslant N, i=1,2,3\}. \tag{18}$$

进一步，有不等式

$$\sum_{j=1}^{p}\ln K_j+\omega\lambda_M<0$$

成立，其中

$$K_j=\max_{(u_1,u_2,u_3)\in\Xi,x\in\Omega}\left\{2\sum_{i=1}^{3}\left[f_{ij}^2+\left(N\frac{\partial f_{1j}}{\partial u_i}\right)^2+\left(N\frac{\partial f_{2j}}{\partial u_i}\right)^2+\left(N\frac{\partial f_{3j}}{\partial u_i}\right)^2\right]\right\},$$

λ_M 是下列矩阵的最大特征值：

$$J=\begin{pmatrix} J_{11} & m_1^{\mathrm{M}}+a_1^{\mathrm{M}} & 0\\ m_1^{\mathrm{M}}+a_1^{\mathrm{M}} & J_{22} & m_2^{\mathrm{M}}+a_2^{\mathrm{M}}\\ 0 & m_2^{\mathrm{M}}+a_2^{\mathrm{M}} & J_{33}\end{pmatrix},$$

这里，

$$J_{11}=2\left[1-\beta-\frac{a_1^{\mathrm{L}}}{(1+N/\beta)^2}\right],$$

$$J_{22}=2\left[\frac{m_1^{\mathrm{M}}}{(1+\beta/N)^2}-b_1^{\mathrm{L}}-\frac{a_2^{\mathrm{L}}}{(1+N/\beta)^2}\right],$$

$$J_{33}=2\left[\frac{m_2^{\mathrm{M}}}{(1+\beta/N)^2}-b_2^{\mathrm{L}}\right].$$

则系统(1)～(5)存在唯一的全局渐近稳定的 ω-周期解.

证明 设$(u_1(t,x),u_2(t,x),u_3(t,x))$和$(u_1^*(t,x),u_2^*(t,x),u_3^*(t,x))$是系统(1)～(5)满足(18)的两个解，定义函数

$$L(t)=\int_\Omega\sum_{i=1}^{3}(u_i(t,x)-u_i^*(t,x))^2\mathrm{d}x,$$

则

$$\begin{aligned}\frac{\mathrm{d}L(t)}{\mathrm{d}t}&=2\int_\Omega\sum_{i=1}^{3}(u_i-u_i^*)\left(\frac{\partial u_i}{\partial t}-\frac{\partial u_i^*}{\partial t}\right)\mathrm{d}x\leqslant\\ &-2d_1\int_\Omega|\nabla(u_1-u_1^*)|^2\mathrm{d}x-2d_2\int_\Omega|\nabla(u_2-u_2^*)|^2\mathrm{d}x-2d_3\int_\Omega|\nabla(u_3-u_3^*)|^2\mathrm{d}x+\\ &2\int_\Omega(u_1-u_1^*)^2\left[1-\beta-\frac{a_1^{\mathrm{L}}}{(1+N/\beta)^2}\right]\mathrm{d}x+\\ &2\int_\Omega(u_3-u_3^*)^2\left[\frac{m_2^{\mathrm{M}}}{(1+\beta/N)^2}-b_2^{\mathrm{L}}\right]\mathrm{d}x+\\ &2\int_\Omega(u_2-u_2^*)^2\left[\frac{m_1^{\mathrm{M}}}{(1+\beta/N)^2}-b_1^{\mathrm{L}}-\frac{a_2^{\mathrm{L}}}{(1+N/\beta)^2}\right]\mathrm{d}x+\\ &2\int_\Omega|(u_1-u_1^*)(u_2-u_2^*)|(m_1^{\mathrm{L}}+a_1^{\mathrm{M}})\mathrm{d}x+\\ &2\int_\Omega|(u_2-u_2^*)(u_3-u_3^*)|(m_2^{\mathrm{M}}+a_2^{\mathrm{M}})\mathrm{d}x\leqslant\\ &\lambda_M\int_\Omega[(u_1-u_1^*)^2+(u_2-u_2^*)+(u_3-u_3^*)^2]\mathrm{d}x.\end{aligned}$$

因此，

$$L(t_{j+1})\leqslant L(t_j+0)\exp(\lambda_M(t_{j+1}-t_j)),$$

$$L(t_{j+1}+0)=\sum_{i=1}^{3}\int_\Omega(u_if_{i(j+1)}(u_1,u_2,u_3)-u_i^*f_{i(j+1)}(u_1^*,u_2^*,u_3^*))^2\mathrm{d}x\leqslant K_{j+1}\exp(\lambda_M(t_{j+1}-t_j))L(t_j+0).$$

同时

$$L(t+\omega)\leqslant K_*L(t)=\prod_{i=1}^{p}K_i\exp(\lambda_M\omega)L(t).$$

由条件知 $K_*<1$，因此

$$L(m\omega+s)\leqslant K_*^{\mathrm{M}}L(s)\to 0,\ m\to\infty.$$

于是，必成立 $\|u_i(t,x)-u_i^*(t,x)|_{L_2}\to 0, t\to\infty, i=1,2,3$. 由引理4可知，系统(1)～(5)的解在空间 C^{1+v} 上有界，并且

$$\sup_{x\in\Omega}|u_i(t,x)-u_i^*(t,x)|\to 0,\ t\to\infty,\ i=1,2,3. \tag{19}$$

根据引理4可知，序列

$$(u_1(k\omega,x,u_{10},u_{20},u_{30}),u_2(k\omega,x,u_{10},u_{20},u_{30}),u_3(k\omega,x,u_{10},u_{20},u_{30}))=w(k\omega,w_0)$$

在空间 $C(\bar{\Omega})\times C(\bar{\Omega})\times C(\bar{\Omega})$ 是紧的，其中 $k\in\mathbb{N}$. 令 $\bar{w}=\lim\limits_{n\to\infty}w(k_n\omega,w_0)$，则 $\bar{w}=w(\omega,w_0)$，并且 $\bar{w}$ 是唯一的. 结合(19)式可知系统(1)～(5)存在唯一渐近稳定的周期解. 】

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（上接第 4 页）

利用最后一列可知 M' 是 M-型模，又由推论 2 知，M' 是 G_C-X-平坦模，再利用中间行及引理 2 可知，M 是 G_C-X-平坦模. 】

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基于 VAR 模型的学前教育发展规模影响因素研究——来自陇南市数据的实证分析

王胜青

(陇南师范高等专科学校 数信学院, 甘肃　陇南 742500)

摘　要: 选取陇南市 2002–2016 年的统计年报数据, 在协整分析的基础上建立学前教育发展规模影响因素的向量自回归模型 (VAR), 然后综合运用格兰杰 (Granger) 因果检验、脉冲响应函数和方差贡献度分解等研究方法, 对经济发展水平、幼儿园校舍面积、专任教师数等因素影响学前教育发展规模的动态效应进行实证分析. 研究结果表明：从长期看, 陇南市学前教育发展规模与经济发展水平、幼儿园校舍面积、专任教师数之间存在长期均衡的协整关系; 经济发展水平、校舍面积对学前教育规模具有正向效应, 专任教师数对学前教育规模起负向效应; 学前教育规模既是经济发展水平的 Granger 原因, 同时也是校舍面积增长的 Granger 原因; 在滞后 2 期内, 经济发展水平和校舍面积均对学前教育规模的脉冲影响具有显著负向效应, 之后沿横轴上下小幅波动并趋于 0, 而专任教师数对学前教育规模增长无显著影响; 在不考虑学前教育发展规模自身贡献率时, 校舍面积对学前教育发展的贡献率最大, 其次是经济发展水平, 而专任教师数对学前教育发展规模贡献率不太明显. 最后根据分析结果提出政策建议.

关键词: 学前教育; 影响因素; VAR 模型; 脉冲响应; 方差分解

学前教育是终身学习的开端, 是国民教育体系的重要组成部分, 是重要的社会公益事业. 办好学前教育, 关系亿万儿童的健康成长, 关系千家万户的切身利益, 关系国家和民族的未来. 改革开放特别是新世纪以来, 我国学前教育取得长足发展, 普及程度逐步提高. 但总体上看, 学前教育仍是各级各类教育中的薄弱环节, 主要表现为教育资源短缺、投入不足, 师资队伍不健全, 体制机制不完善, 城乡区域发展不平衡, 一些地方 “入园难” 问题突出. 对此,《国务院关于当前发展学前教育的若干意见》(国发 [2010]41 号) 提出要 “把发展学前教育摆在更加重要的位置”, 要求各省 (区、市) 以县为单位编制实施学前教育三年行动计划. 同时, 为支持各地实施好学前教育三年行动计划, 国家启动实施了一系列重大项目, 重点支持中西部地区发展农村学前教育. 截止 2016 年底, 甘肃省已实施了二轮次的三年行动计划. 2011 年至 2016 年, 陇南市共实施省级财政学前教育幼儿园建设项目 383 个, 累计投资 48042 万元, 其中省级财政下达资金 42838 万元, 市县区自筹资金 5104 万元, 完成建筑面积 22.1 万平方米, 设置班级 1822 个, 新增幼儿入园园位 5.5 万个, 在园学生数已达到 108774 人, 是 2010 年在园

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资助项目: 甘肃省教育科学 “十三五” 规划 2016 年度课题：陇南学前教育发展水平研究——基于动态面板数据的实证分析 (GS[2016]GHB0185)

学生数 23099 人的 4.7 倍, 三年毛入园率达到了 92.12%. 第二期学前教育三年行动计划完成之后, 校舍面积已通过新建、改建而扩充, 专任教师得到补给, "入园难" 问题已基本解决. 今后学前教育如何均衡发展, 是本文研究的主要问题.

目前, 国内学者对学前教育发展规模、经济发展水平、校舍面积、专任教师数之间的均衡关系进行实证研究的文献基本空白, 运用计量经济学模型来研究学前教育发展水平的文献也不多, 对学前教育的研究大都局限在通过对现象的简单分析并提出建议方面. 陇南市作为秦巴山区集中连片的贫困地区, 研究其学前教育发展规模的影响因素及均衡关系具有一定的普遍意义. 因此, 本文尝试对学前教育发展规模、经济发展水平、幼儿园校舍面积、专任教师数之间的均衡关系进行计量分析, 并提出一些合理化的政策建议.

1 模型与方法

向量自回归 (VAR) 是基于数据的统计性质建立模型, 它是把系统中每一个内生变量作为系统中所有内生变量的滞后值的函数来构造模型, 从而将单变量自回归模型推广到由多元时间序列变量组成的 "向量" 自回归模型, 通常用来估计相互联系的时间序列系统以及分析随机扰动对变量系统的动态关系, 不需要提前设定任何约束条件.

其数学表达式为:

$$y_t = A_1 y_{t-1} + \cdots + A_p y_{t-p} + Bx_t + \varepsilon_t \quad . \quad t = 1, 2, \cdots, T \tag{1}$$

式中, y_t 是 k 维内生变量向量, x_t 是 d 维外生变量向量, p 是滞后阶数, 样本个数为 T; $y_{t-1}, \cdots, y_{t-p}$ 表示 y_t 的滞后期; $k \times k$ 维矩阵 $A_1, \cdots, A_p$ 和 $k \times d$ 维矩阵 B 是待估系数矩阵; ε_t 是 k 维随机扰动向量, 它们相互之间可以同期相关, 但不与自己的滞后值相关及不与等式右边的变量相关. VAR 模型的矩阵形式为:

$$\begin{bmatrix} y_{1t} \\ y_{2t} \\ \vdots \\ y_{kt} \end{bmatrix} = A_1 \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \\ \vdots \\ y_{k,t-1} \end{bmatrix} + A_2 \begin{bmatrix} y_{1,t-2} \\ y_{2,t-2} \\ \vdots \\ y_{k,t-2} \end{bmatrix} + \cdots + BX_t + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \vdots \\ \varepsilon_{kt} \end{bmatrix} \tag{2}$$

即含有 k 个时间序列变量的 $VAR(p)$ 模型由 k 个方程组成. 式中, $y_{1t}, \cdots, y_{kt}$ 作为内生变量, 可以同期相关, 而 $y_{1,t-1}, \cdots, y_{k,t-p}$ 作为滞后变量均在等号右边, 因此不会出现同期相关问题, OLS 仍然是有效的 [1].

该模型不关心方程回归系数是否显著, 检验重点是模型整体的稳定性水平, 只有在 VAR 系统稳定的基础上, 才能利用脉冲响应和方差分解来研究随机扰动对变量系统的动态冲击.

2 变量选取与数据说明

采用陇南市 2002—2016 年统计年报数据 (见表 1、表 2), 利用计量经济分析软件 Eviews8.0, 选取地方经济发展水平、幼儿园校舍面积、专任教师数为学前教育发展规模的影响因素, 建立 VAR 模型, 运用脉冲响应函数与方差分解的计量分析方法, 对影响陇南市学前教育发展规模的各因素动态关系进行实证分析, 探讨变量之间的效应关系.

考虑到数据的可获得性和研究需要, 本文选取学前教育在校学生人数 (STU)、人均 GDP (GDP)、校舍总面积 (HOU)、专任教师数 (TEA) 作为研究变量. 为消除物价变动对 GDP 的

影响, 采用商品零售价格指数对 GDP 数据进行平减, 将样本期内 GDP 的数据调整为上年的不变价格.

表 1

年度	学生数 (人)	专任教师 (人)	校舍面积 (平方米)	人均 GDP(元)
2002	19403	458	28730	1778.022474
2003	20353	500	31736	1936.899456
2004	18344	482	30614	2298.158667
2005	22070	461	31922	2737.328377
2006	20189	461	34648	3334.71344
2007	16060	479	32806	3829.283026
2008	22559	530	28798	4112.986088
2009	20659	553	30258	4928.743054
2010	23099	584	39127	5809.637799
2011	34702	580	41643	6644.707913
2012	46237	674	58956	7863.429906
2013	59528	989	161440	8684.640894
2014	67857	1038	145916	10108.31913
2015	78263	1298	247439	10929.76759
2016	108774	1658	326263	11696.38523

表 2

年度	总人口 (万)	GDP(万)	商品价格指数	价调 GDP(万)
2002	266.09	473114	100	473114
2003	267.61	523517	101	518333.6634
2004	268.81	633830	102.6	617768.0312
2005	268.59	741836	100.9	735219.0287
2006	270.75	929057	102.9	902873.6638
2007	275.19	1118061	106.1	1053780.396
2008	279.18	1216011	105.9	1148263.456
2009	282.3	1423386	102.3	1391384.164
2010	281.77	1694276	103.5	1636981.643
2011	281.46	1976822	105.7	1870219.489
2012	280.64	2259756	102.4	2206792.969
2013	282.77	2495048	101.6	2455755.906
2014	283.23	2894472	101.1	2862979.228
2015	285.76	3151400	100.9	3123290.387
2016	287.81	3400000	101	3366336.634

同时, 为消除数据中存在的异方差, 分别对这些变量 (STU,GDP,HOU,TEA) 取自然对数,

分别记为 LNSTU、LNGDP、LNHOU、LNTEA. 这样, 不仅能消除时间序列中的异方差现象, 而且使变量变为弹性变量.

3 实证分析

3.1 变量的平稳性检验

在建立 VAR 模型之前, 需先对各个时间序列做平稳性检验. 本文运用 ADF 检验法对变量 LNSTU、LNGDP、LNHOU、LNTEA 进行单位根检验, 检验结果见表 3.

表 3 ADF 单位根检验结果

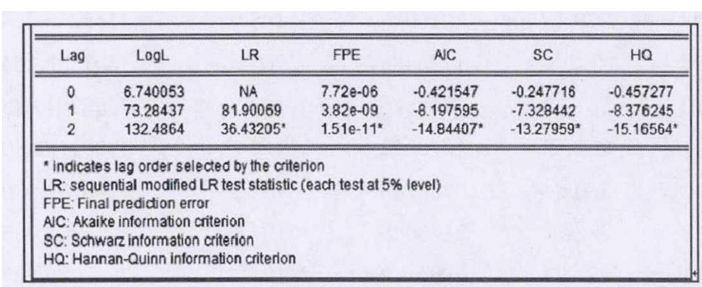
变量	ADF 检验值	临界值 (1%)	临界值 (5%)	临界值 (10%)	结论
LNSTU	−1.117266	−4.800080	−3.791172	−3.342253	不平稳
dLNSTU	−4.322646	−4.886426	−3.828975	−3.362984	平稳 **
LNGDP	−0.642875	−4.800080	−3.791172	−3.342253	不平稳
dLNGDP	−3.981087	−4.886426	−3.828975	−3.362984	平稳 **
LNHOU	−2.367384	−5.124875	−3.933364	−3.420030	不平稳
dLNHOU	−4.687783	−4.886426	−3.828975	−3.362984	平稳 **
LNTEA	−0.113088	−4.800080	−3.791172	−3.342253	不平稳
dLNTEA	−4.385428	−4.886426	−3.828975	−3.362984	平稳 **

注：表 3 中的 d 表示一阶差分.** 表示在 5%水平下显著.

由表 3 可知: LNSTU、LNGDP、LNHOU、LNTEA 等 4 个变量的原序列均存在单位根, 均为非平稳序列; 而经过一阶差分后, 4 个差分序列都通过了 5% 显著性水平的平稳性检验, 说明这些变量的差分序列是平稳的 [2], 均属于一阶单整序列, 即 I(1) 过程.

3.2 确定 VAR 模型的滞后阶数

为了确保模型拥有合理的自由度和较强的解释力, VAR 模型需要确定最佳滞后期, 一般可以根据赤池信息准则 (AIC) 和施瓦茨准则 (SC) 取值最小的原则来确定 VAR 模型的滞后阶数. 结果显示, AIC 准则和 SC 准则的值在滞后期为 2 时同时最小, 故 VAR 模型的最佳滞后阶数 P 为 2, 此模型为 VAR(2) 模型. 不同滞后期各准则数值的计算结果详见图 1(注：* 表示各准则确定的最佳滞后期).



Lag	LogL	LR	FPE	AIC	SC	HQ
0	6.740053	NA	7.72e-06	-0.421547	-0.247716	-0.457277
1	73.28437	81.90069	3.82e-09	-8.197595	-7.328442	-8.376245
2	132.4864	36.43205*	1.51e-11*	-14.84407*	-13.27959*	-15.16564*

* indicates lag order selected by the criterion
LR: sequential modified LR test statistic (each test at 5% level)
FPE: Final prediction error
AIC: Akaike information criterion
SC: Schwarz information criterion
HQ: Hannan-Quinn information criterion

图 1 不同滞后期各准则数值的计算结果

3.3 VAR 模型的平稳性检验

用 Eviews8.0 计算所构建的 VAR(2) 模型所有根的倒数的模, 来检测 VAR 模型的稳定性.

对于滞后期为 2 且有 4 个内生变量的 VAR 模型来说, 其 AR 特征多项式共有 2×4 即 8 个根. 经检测发现, 没有根位于单位圆外, 即这些根的倒数的模都小于 1, 这表明所估计的 VAR(2) 模型是稳定的, 可以进行后续的分析, 并且后续分析结果是合理的. 检验结果见图 2.

Hypothesized No. of CE(s)	Eigenvalue	Trace Statistic	0.05 Critical Value	Prob.**
None *	0.999539	137.7112	47.85613	0.0000
At most 1 *	0.812841	37.84054	29.79707	0.0048
At most 2 *	0.558449	16.05521	15.49471	0.0412
At most 3 *	0.341344	5.428204	3.841466	0.0198

图 2 AR 特征多项式根检验结果

3.4 协整检验

为了检验变量 LNSTU、LNGDP、LNHOU、LNTEA 之间是否存在长期稳定的均衡关系, 采用 Johansen 协整检验. 进行协整检验选择的滞后阶数应该等于无约束的 VAR 模型的最优滞后阶数减 1, 即协整检验的最后滞后阶数为 1[3]. 协整检验结果见图 3.

Hypothesized No. of CE(s)	Eigenvalue	Trace Statistic	0.05 Critical Value	Prob.**
None *	0.999539	137.7112	47.85613	0.0000
At most 1 *	0.812841	37.84054	29.79707	0.0048
At most 2 *	0.558449	16.05521	15.49471	0.0412
At most 3 *	0.341344	5.428204	3.841466	0.0198

图 3 Johansen 协整检验结果

根据迹统计量 (Trace Statistic) 结果, 以检验水平 0.05 来看, 均拒绝了原假设. 因此, 在 5%的显著水平下变量 LNSTU、LNGDP、LNHOU、LNTEA 之间存在协整关系, 并有 4 个协整方程. 一般选取第一个协整向量作为研究变量之间存在均衡关系的协整向量. 检验结果见图 4.

1 Cointegrating Equation(s):		Log likelihood	103.8973	
Normalized cointegrating coefficients (standard error in parentheses)				
LNSTU	LNGDP	LNHOU	LNTEA	C
1.000000	-0.825629	-1.669778	2.956906	-5.027376
	(0.00816)	(0.01385)	(0.03464)	(0.06003)

图 4 标准化的协整系数

变量系统的标准化协整向量为 $(1.0000, -0.825629, -1.669778, 2.956906, -5.027376)$, 对应的协整方程为:

$$LNSTU = 0.825629 * LNGDP + 1.669778 * LNHOU - 2.956906 * LNTEA + 5.027376. \quad (3)$$

根据以上协整方程, 可以得出经济发展水平 (LNGDP)、校舍面积 (LNHOU)、专任教师数 (LNTEA) 与学前教育规模 (LNSTU) 存在长期协整关系. 并且从它们之间的协整方程可以看出: 经济发展水平的长期弹性是 0.825629, 校舍面积的长期弹性是 1.669778, 专任教师数的长

期弹性是负向 2.956906. 也就是说, 从长期均衡关系来看, 经济发展水平每增长 1%, 学前教育规模增长 0.825629%; 校舍面积每增长 1%, 学前教育规模增长 1.669778%; 而专任教师数每增长 1%, 学前教育规模反而降低 2.956906%[4]. 根据各指标的弹性值可以判定, 经济发展水平和校舍面积对学前教育规模有促进作用, 专任教师数对学前教育规模有抑制作用.

3.5 Granger 因果检验

格兰杰 (Granger) 因果检验是用于检验变量之间的时间先后顺序, 并不表示真正存在因果关系, 是否存在因果关系需要根据理论、经验和模型进行综合判断. 变量间的 Granger 因果关系检验法是由计量经济学家 Granger 于 1969 年利用分布滞后概念提出的. Granger 因果关系检验的基本思想是: 在时间序列 X 和 Z 消除了趋势之后, 如果利用过去的 X 和 Z 的值一起对 Z 进行预测, 比单独用 Z 的过去值预测的效果更好的话, 序列 X 和 Z 之间存在因果关系, 则称 X 是 Z 的 Granger 原因, 记为 X → Z[5]. 由于经济发展水平 (LNGDP)、校舍面积 (LNHOU)、专任教师数 (LNTEA) 与学前教育规模 (LNSTU) 存在长期均衡关系, 因此可以进行 Granger 因果检验. 选取滞后阶数为 2, 检验水平 0.05, 检验结果见图 5.

Null Hypothesis:	Obs	F-Statistic	Prob.
LNGDP does not Granger Cause LNSTU	13	2.22079	0.1709
LNSTU does not Granger Cause LNGDP		4.64388	0.0459
LNHOU does not Granger Cause LNSTU	13	0.02815	0.9723
LNSTU does not Granger Cause LNHOU		28.9490	0.0002
LNTEA does not Granger Cause LNSTU	13	0.48686	0.6316
LNSTU does not Granger Cause LNTEA		2.49295	0.1440
LNHOU does not Granger Cause LNGDP	13	1.04634	0.3948
LNGDP does not Granger Cause LNHOU		2.14467	0.1796
LNTEA does not Granger Cause LNGDP	13	0.27682	0.7652
LNGDP does not Granger Cause LNTEA		2.02401	0.1944
LNTEA does not Granger Cause LNHOU	13	0.84655	0.4640
LNHOU does not Granger Cause LNTEA		0.09549	0.9099

图 5 Granger 因果关系检验结果

由图 5 知, 滞后期为 2 时, 在 5%的显著性水平下, 学前教育规模 (LNSTU) 与经济发展水平 (LNGDP) 具有单向 Granger 因果关系, 即学前教育规模 (LNSTU) 是经济发展水平 (LNGDP) 的 Granger 原因; 学前教育规模 (LNSTU) 与校舍面积 (LNHOU) 具有单向 Granger 因果关系, 即学前教育规模 (LNSTU) 是校舍面积 (LNHOU) 的 Granger 原因; 学前教育规模 (LNSTU) 与专任教师数 (LNTEA) 没有 Granger 因果关系. 另外, 校舍面积 (LNHOU) 与经济发展水平 (LNGDP)、专任教师数 (LNTEA) 与经济发展水平 (LNGDP)、专任教师数 (LNTEA) 与校舍面积 (LNHOU) 之间也没有 Granger 因果关系. 这说明在短期内, 经济发展水平、校舍面积、专任教师数的前期变化在统计上不能有效地预测学前教育发展规模. 原因可能是受陇南地区山大沟深、乡村农户居住分散等因素的制约, 地区经济水平提高的短期效应和学前教育资源的集中配备对学前教育规模的影响并不明显.

3.6 脉冲响应函数分析 [5]

脉冲响应函数分析方法可以用来描述一个内生变量对由误差项所带来的冲击的反应, 即

在随机误差项上施加一个标准差大小的冲击后，对内生变量的当期值和未来值所产生的影响程度.

从上述可知，变量 LNSTU、LNGDP、LNHOU、LNTEA 建立的 VAR 模型是稳定有效的，所以在此基础上，我们可以利用 Eviews8.0 软件，通过设置脉冲冲击变量、追踪期数以及输出方式等信息，即可得到各变量一个标准差新息 (Innovation) 的冲击对学前教育规模的脉冲响应函数图.

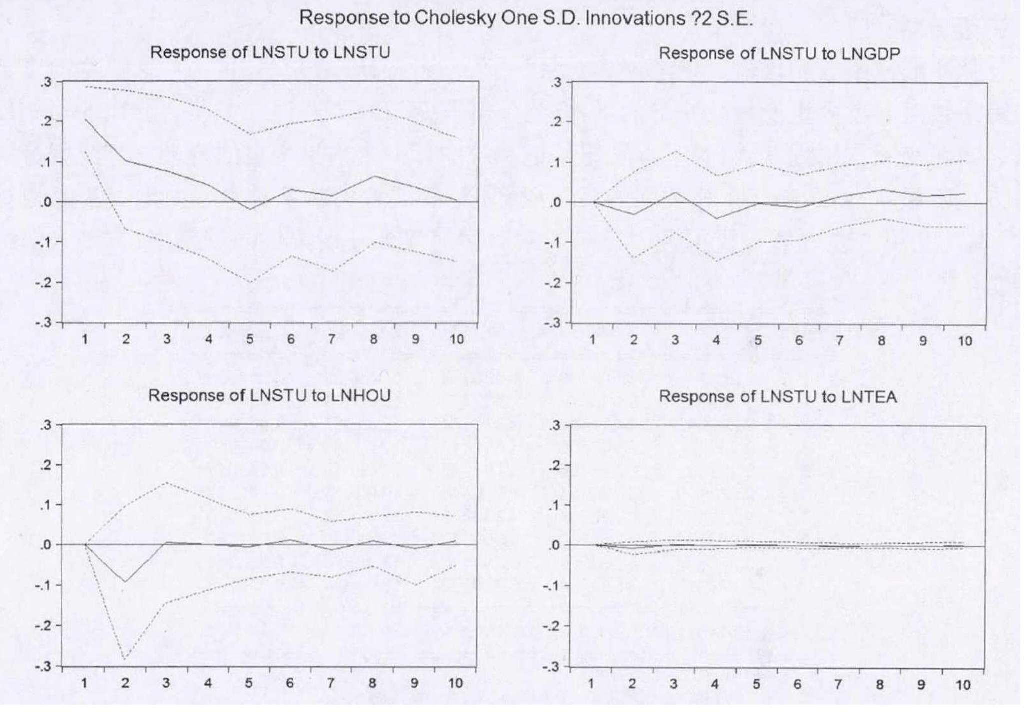


图 6　学前教育规模对各变量冲击的响应

图 6 中，分别显示学前教育规模对自身及人均 GDP、校舍面积、专任教师数等变量冲击的响应，横轴表示脉冲响应的追踪期数，纵轴表示 LNSTU 对 LNSTU、LNGDP、LNHOU、LNTEA 的反应程度，实线表示脉冲响应函数，虚线表示正负两倍的标准差偏离带 (±2S.E). 从图中可以看出，当学前教育规模 (LNSTU) 对自身经过一个标准差新息的正向冲击后，学前教育规模增长协调性初期反应较大，并达到正向最大值 (0.20)，之后各期影响快速下降，但还是正向影响，在第 5 期变为负向影响，达到最低值 (−0.02)，随后影响又转为正向并有小幅波动，上升至第 8 期 (0.06) 之后又迅速向 0 轴逼近. 总体而言，学前教育规模对自身的冲击影响力度最大，在滞后 10 期之内呈正向反应，当期表现最显著，第 5 期之后趋于稳定；当经济发展给学前教育规模一个标准差新息的正向冲击后，学前教育规模的脉冲响应在第 1 期为 0，第 2 期为负向反应 (−0.03)，第 3 期又达到正向值 (0.03)，第 4 期降至负向最小值 (−0.04)，之后呈小幅度的上下波动趋势，到第 6 期之后呈正向反应并趋于稳定. 这说明，经济发展对学前教育规模的后期影响时限长，经济发展水平是保持学前教育发展规模稳步增长的重要因素；当校舍面积增长给学前教育规模一个标准差新息的正向冲击后，学前教育规模的脉冲响应在第 1 期为

0, 然后有显著的负向反应, 第 2 期达到负向最小值 (-0.09), 第 3 期之后呈现向零效应收敛的迹象在 0 附近保持稳定, 说明短期内校舍面积的无序扩充对学前教育规模增长协调性具有负面影响, 但很快会趋于稳定, 在滞后十期之内增加校舍面积和扩大学生规模在系统中将保持长期均衡关系; 当专任教师数给学前教育规模一个标准差新息的正向冲击后, 脉冲响应曲线几乎和横轴重合, 纵轴正向值基本保持为常数 (0.01), 说明专任教师数在滞后各期内对学前教育规模增长无显著影响, 随着时间的推移, 增加专任教师数和扩大学生规模在系统中将保持长期均衡关系.

3.7 方差分解 [5]

与脉冲响应函数相比较, 方差分解提供了另外一种描述系统动态的方法. 脉冲响应函数是追踪系统对一个内生变量的冲击效果, 方差分解以变量的预测误差方差百分比的形式反映变量之间的交互作用程度, 它的基本思想是把系统中每一个内生变量的变动按其成因分解为与各方程随机扰动项 (新息) 相关联的各组成部分, 以了解各新息对模型内生变量的相对重要性. 本文利用方差分解分析了各个影响因素对学前教育发展规模的贡献率, 方差分解结果见图 7.

Period	S.E.	LNSTU	LNGDP	LNHOU	LNTEA
1	0.204785	100.0000	0.000000	0.000000	0.000000
2	0.249598	84.38414	1.512407	14.00501	0.098449
3	0.262942	84.87444	2.376444	12.65627	0.092844
4	0.269131	83.41763	4.411641	12.08191	0.088812
5	0.269802	83.45355	4.393182	12.06018	0.093088
6	0.271909	83.36794	4.515648	12.02411	0.092307
7	0.273140	82.93288	4.853822	12.12018	0.093119
8	0.282631	82.54862	5.950663	11.41346	0.087260
9	0.285624	82.39009	6.235410	11.28876	0.085749
10	0.286866	81.69264	6.916930	11.30014	0.090287

Cholesky Ordering: LNSTU LNGDP LNHOU LNTEA

图 7 学前教育发展规模方差分解结果

方差分解结果表明, 对学前教育规模发展变化贡献率最大的还是自身因素的变化, 虽然学前教育规模对自身的贡献率在各滞后期呈递减趋势, 其中较为明显的是第二期贡献率比第一期下降了 15.62 个百分点, 但第十期其贡献率仍然达到 81.69%, 依旧起着主要作用, 这和我国实施学前教育两轮三年行动计划及二孩生育政策有关, 即我国学前教育规模的扩张主要靠政府行政力量来推动. 其次, 对学前教育规模增长较为明显的影响因素是幼儿园校舍面积, 第 2 期就达到了最大值 14%, 之后在 12%上下轻微波动, 第十期其贡献率仍然达到了 11.3%, 这与陇南实施省级财政学前教育幼儿园建设项目有关, 短期内增加校舍面积, 解决入园难问题, 从而扩充了学前教育的规模. 再次是人均 GDP, 随着时间的推移其贡献率整体呈上升趋势, 第十期达到了 6.92%, 这说明从长远来看经济增长仍然是影响学前教育发展规模增长的不变因素. 对学前教育规模增长贡献率最低的是专任教师数, 从滞后各期来看, 其贡献率最大不足 0.1%, 而且长期稳定在 0.09%左右.

4 结论

本文选取陇南市 2002-2016 年的统计年报数据, 在协整分析的基础上建立学前教育发展

规模影响因素的向量自回归模型 (VAR), 然后综合运用格兰杰 (Granger) 因果检验、脉冲响应函数和方差贡献度分解等研究方法, 对经济发展水平、幼儿园校舍面积、专任教师数等因素影响学前教育发展规模的动态效应进行实证分析. 研究结果表明: 从长期看, 陇南市学前教育发展规模与经济发展水平、幼儿园校舍面积、专任教师数之间存在长期均衡的协整关系; 经济发展水平、校舍面积对学前教育规模具有正向效应, 专任教师数对学前教育规模起负向效应; 从短期看, 学前教育规模既是经济发展水平的 Granger 原因, 同时也是校舍面积增长的 Granger 原因, 但经济发展水平、校舍面积、专任教师数的前期变化在统计上不能有效地预测学前教育发展规模. 在滞后 2 期内, 经济发展水平和校舍面积均对学前教育规模的脉冲影响具有显著负向效应, 之后沿横轴上下小幅波动并趋于 0, 而专任教师数对学前教育规模增长无显著影响. 这说明, 未来两年内学前教育规模不会有太大的增长, 当提高经济发展水平、增加校舍面积和专任教师数时对学前教育规模的影响不明显, 但随着时间推移, 系统将保持长期均衡关系; 从贡献率层面来看, 在不考虑学前教育发展规模自身贡献率时, 校舍面积对学前教育发展的贡献率最大, 其次是经济发展水平, 而专任教师数对学前教育发展规模贡献率不太明显. 这说明陇南市在着力解决适龄儿童 “入园难” 问题上, 增加政府投入新建、改建、扩建了一批安全、适用的幼儿园, 从而扩大了幼儿园校舍面积, 满足了适龄儿童有学上, 对学前教育发展的贡献率最大符合发展规律.

综上所述, 陇南市学前教育发展规模与经济发展水平、幼儿园校舍面积、专任教师数之间存在长期均衡的协整关系. 当政府完成了学前教育两轮三年行动计划之后, 校舍面积得到了快速增长, 专任教师得到了足够补充, 学前教育三年毛入园率达到了 92.12%, 学前教育规模基本趋于稳定. 由于陇南属于秦巴山区集中连片贫困区, 是全省贫困人口最多的地区之一, 经济落后、山大沟深、乡村农户居住分散、村级幼儿园短缺是制约学前教育发展的主要因素. 今后, 地方政府要继续保持学前教育的投入水平, 并根据二孩生育政策的放开, 面临适龄儿童规模的稳步增长, 适当扩充校舍面积, 重点放在村级幼儿园的改扩建上, 并按省定师生比标准补充专任教师数. 同时, 从长期来看, 还是要大力发展地方经济, 通过提高居民支付能力来增加对学前教育的需求, 扩大学前教育规模. 针对城乡经济水平差异大、学前教育发展不均衡的问题, 地方政府应制订统筹城乡教育发展规划, 宏观调控城乡教育资源的配置. 一方面, 要加大对农村园、薄弱园的扶持力度, 充分发挥经济为教育服务的功能, 公共财政经费向农村及乡镇学前教育倾斜, 重点是建立长效、规范的保障制度, 用优质教学条件和学前教育服务留住幼儿; 另一方面, 随着城镇化进程的推进, 对于县城学前教育, 面对进城务工人员随迁子女人数剧增的趋势, 要科学测算适龄儿童, 积极扩大学前教育资源量, 满足随迁儿童的入园需求. 今后, 在保证适龄儿童有园上的同时, 还要着力解决上好园的问题.

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Study on the Influencing Factors of Preschool Education Development Scale Based on the Var Model — An empirical analysis of Longnan city's data

WANG Sheng-qing

(College of Mathematics and Information Technology, Longnan Teachers College, Longnan 742500, China)

Abstract: This paper, based on cointegration analysis, selects Longnan city's annual report data from 2002 to 2016 to establish VAR model of influencing factors of preschool education development scale, and analyzes effect of economic development level, kindergarten school area and number of full-time teachers on the dynamic effect of influencing preschool education development scale with the comprehensive use of Granger causality test, impulse response function and variance contribution decomposition. The results show that in the long run, there is a long-term cointegration relationship among preschool education development scale, economic development level, kindergarten school area and number of full-time teachers. Preschool education development scale is Granger cause of not only economic development level, but also school area growth. Economic development level and school area play a role in impulsive effects of preschool education development scale and have a significant negative effect, then change slightly at the transverse axis and tend to zero, but number of full-time teachers no significant effect in the lag two phase. Without considering the contribution rate of preschool education development scale, school area has largest contribution ratio on preschool education development, and follows by economic development level, while number of full-time teachers no obvious effect. Based on the result of the above analysis, we put forward some suggestions.

Keywords: preschool education; influencing factor; VAR model; impulse response; variance decomposition

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复形的 W-Gorenstein 预覆盖

崔俊峰

（陇南师范高等专科学校 数学系，甘肃 成县 742500）

摘要：令 W 是 R-模的自正交类. 证明任意具有有限 W-Gorenstein 分解维数的 R-复形 X 都有 W-Gorenstein 预覆盖 $f:G \to X$，且 f 是满的拟同构. 作为应用，将 Holm 关于模的 Gorenstein 投射预覆盖的结论推广到了复形.

关键词：W-Gorenstein 复形；自正交类；预覆盖

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W-Gorenstein precover of complexes

CUI Jun-feng

(Department of Mathematics, Longnan Teachers College, Cheng County 742500, China)

Abstract: Let W be a self-orthogonal class of R-modules. It is proved that for any R-complex X with finite W-Gorenstein resolution dimension, there exists a W-Gorenstein precover $f:G \to X$, where f is an epic quasi-isomorphism. As an application, we generalize the Holm's result on Gorenstein projective precover of modules onto the complexes.

Key words: W-Gorenstein complex; self-orthogonal class; precover

设 X 是阿贝尔范畴 A 中的一个对象，B 是 A 中的一个对象类. 设 $B \in$ B. 称同态 $f:B \to X$ 是 X 的 B-预覆盖，若对于任意 $B' \in$ B，序列 $\mathrm{Hom}(B', B) \to \mathrm{Hom}(B', X) \to 0$ 是正合的. 等价地，对于任意 $B' \in$ B，任意同态 $g:B' \to X$，存在同态 $h:B' \to B$ 使得图 1 可交换.

$$\begin{tikzcd} & B' \arrow[dl, dotted, "h"'] \arrow[d, "g"] \\ B \arrow[r, "f"'] & X \end{tikzcd}$$

图 1　预覆盖图

Fig. 1　Precover diagram

若 B-预覆盖存在，则可以通过 B 中的对象去逼近范畴 A 中的任意对象，进而通过 B 的性质研究范畴 A. 特别是在模的复形范畴中，（预）覆盖的研究受到了许多学者的广泛关注[1-2].

本文研究了复形的 W-Gorenstein 预覆盖，其中 W 是 R-模的自正交类. 主要结论为

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作者简介：崔俊峰(1978-)，男，甘肃陇西人，讲师.

定理 1　设复形 X 的 W-Gorenstein 分解维数为 n. 则有 W-Gorenstein 预覆盖 $f:G \to X$，其中 f 是满的拟同构，且 $\mathrm{Ker}(f)$ 的 W 分解维数为 $n-1$.

Enochs 和 Garcia Rozas[3] 研究交换局部 Gorenstein 环上复形的 Gorenstein 投射（预）覆盖. 特别地，令 W 为所有投射模的类，则 W-Gorenstein 投射复形正是 Gorenstein 投射复形. 此时，W-Gorenstein 分解维数即 Gorenstein 投射维数，记为“Gpd”. 由于任意模可以看作是集中在 0 层次的复形，下述结论将 Holm 在文献［4］中关于模的 Gorenstein 投射预覆盖的结论推广到了复形. 用“Gpd”和“pd”分别表示相对于 Gorenstein 投射和投射对象的分解维数. 模的(Gorenstein)投射维数的两种推广，即通过分解定义的(Gorenstein)投射维数和投射维数，与通过复形的拟同构定义的(Gorenstein)投射维数和投射维数[5-6]，对于复形而言是不同的.

推论 1　设 R 是任意结合环，X 是 R-复形. 若 $\mathrm{Gpd}(X)=n$，则有 Gorenstein 投射预覆盖 $f:G \to X$，其中：f 是满的拟同构，且 $\mathrm{pd}(\mathrm{Ker}(f))=n-1$. 特别地，若模 M 具有有限 Gorenstein 投射维数，则存在 Gorenstein 投射预覆盖 $f:H \to M$，其中：f 是满射且

$\mathrm{pd}(\mathrm{Ker}(f))=\mathrm{Gpd}(M)-1$.

假设 R 是交换 Noether 环，C 是有限生成的 R-模. 称 C 是半对偶模，如果对于任意 $i\geqslant 1$，$\mathrm{Ext}_R^i(C,C)=0$，并且自然的态射 $R\to\mathrm{Hom}_R(C,C)$ 是同构. 称 R-模 M 是 C-投射的，如果存在投射模 P 使得 $M\cong C\otimes_R P$. 令 P_C 表示所有 C-投射模的类. 由文献[7]可知 P_C 是一个自正交类. 由于在文献[8]中称 P_C-Gorentein 模为 C-Gorenstein 投射模，相应地称 $\widetilde{\mathrm{P}_C}$-Gorentein 复形为 C-Gorenstein 投射复形. 进而，称 R-复形 X 的 $\widetilde{\mathrm{P}_C}$-Gorentein 分解维数为 C-Gorenstein 投射维数，记为 C-$\mathrm{Gpd}(X)$. 记号 C-$\mathrm{pd}(X)$ 表示 X 的 $\widetilde{\mathrm{P}_C}$-分解维数，称为 X 的 C-投射维数.

推论 2　设 R 是具有半对偶模 C 的交换 Noether 环，X 是 R-复形. 若 C-$\mathrm{Gpd}(X)=n$，则有 C-Gorenstein投射预覆盖 $f:G\to X$，其中 f 是满的拟同构，且 C-$\mathrm{pd}(\mathrm{Ker}(f))=n-1$.

注 1　在文献[7]注 2.3 中列举了许多自正交模类的例子. 因此，由定理 1，可以得到诸如复形的 V-Gorenstein 投射预覆盖，Ω-Gorenstein 投射预覆盖等结论. 关于 V-Gorenstein 投射和 Ω-Gorenstein 模，详见文献[8].

1　预备知识

定义 1[7]　令 W 表示一个 R-模的类. 称 W 是自正交的，如果它满足下面的条件：

$$\mathrm{Ext}_R^i(W,W')=0\qquad \forall W,W'\in \mathrm{W},\forall i\geqslant 1$$

用 W 表示一个关于扩张，有限直和以及直和因子封闭的 R-模的自正交类. 用记号 $\widetilde{\mathrm{W}}$ 和 $\widetilde{\mathrm{WG}}$ 分别表示所有 W 复形的类和所有 W-Gorenstein 复形的类.

R-复形 X 通常表示为 R-模的序列 $\cdots\to X_1\xrightarrow{d_1^X}X_0\xrightarrow{d_0^X}X_{-1}\xrightarrow{d_{-1}^X}X_{-2}\to\cdots$ 其中：对于任意的 n，有 $d_{n-1}^X d_n^X=0$.

定义 2　称 R-复形 X 是 W 复形，如果 X 是正合复形，并且对于任意 $n\in\mathbf{Z}$，$\mathrm{Ker}(d_n^X)\in\mathrm{W}$.

记号 $\mathrm{Hom}_{C(R)}(-,-)$ 表示 R-复形的范畴 $\mathrm{C}(R)$ 中的态射集，$\mathrm{Ext}_{C(R)}(-,-)$ 表示 $\mathrm{Hom}_{C(R)}(-,-)$ 在范畴 $\mathrm{C}(R)$ 中的导出函子.

定义 3[9]　称 R-复形 X 是 $\widetilde{\mathrm{W}}$-Gorenstein 复形，如果存在 $\widetilde{\mathrm{W}}$ 中复形的正合序列

$$\mathbf{W}=\cdots\to V_1\to V_0\to V_{-1}\to V_{-2}\to\cdots$$

使 $X=\mathrm{Ker}(V_0\to V_{-1})$，且 $\mathbf{W}$ 同时是 $\mathrm{Hom}_{C(R)}(\widetilde{\mathrm{W}},-)$ 和 $\mathrm{Hom}_{C(R)}(-,\widetilde{\mathrm{W}})$ 正合的.

令 A 是一个 Abel 范畴，X 是 A 的全子加法范畴. 设 $M\in$ A. 称正合序列 $\cdots\to X_1\to X_0\to M\to 0$ 为 M 的 X 分解，其中：$X_i\in$ X, $\forall i\geqslant 0$. 定义 M 的 X 分解维数（记为 X-resdim(M)），为 M 的所有 X 分解的最小长度，即

X-resdim$(M)=\inf\{n\in\mathbf{Z}\mid$ 存在 M 的 X 分解 $0\to X_n\to X_{n-1}\to\cdots\to X_0\to M\to 0\}$

特别地，令 A 为 R-复形的范畴，X 分别为 $\widetilde{\mathrm{W}}$ 和 $\widetilde{\mathrm{WG}}$. 则有复形 M 的 W 和 W-Gorenstein 分解维数的概念，分别记为 $\widetilde{\mathrm{W}}$-resdim(M) 和 $\widetilde{\mathrm{WG}}$-resdim(M). 文献[10]中定理 1 讨论了复形的 W-Gorenstein 分解维数与模的 W-Gorenstein 分解维数之间的关系.

2　定理 1 的证明

引理 1　设 X,Y 是 R-复形. 若 X 是 W-Gorenstein 复形，Y 具有有限的 $\widetilde{\mathrm{W}}$ 分解维数，则对于任意 $i\geqslant 1$，有 $\mathrm{Ext}_{C(R)}^i(X,Y)=0$.

证明　由 W-Gorenstein 复形的定义可知，对于任意 $W\in\widetilde{\mathrm{W}}$ 及 $i\geqslant 1$，有 $\mathrm{Ext}_{C(R)}^i(W,X)=\mathrm{Ext}_{C(R)}^i(X,W)=0$. 设 K 的 $\widetilde{\mathrm{W}}$ 分解维数为 n，则有下列正合序列

$$0\to W_n\to W_{n-1}\to\cdots\to W_1\to W_0\to Y\to 0$$

其中：$W_k\in\widetilde{\mathrm{W}}$, $k=0,1,\cdots,n$. 考虑短正合序列 $0\to K_1\to W_0\to Y\to 0$. 由

$$0=\mathrm{Ext}_{C(R)}^i(X,W_0)\to\mathrm{Ext}_{C(R)}^i(X,Y)\to \mathrm{Ext}_{C(R)}^{i+1}(X,K_1)\to\mathrm{Ext}_{C(R)}^{i+1}(X,W_0)=0$$

可得 $\mathrm{Ext}_{C(R)}^i(X,Y)\cong\mathrm{Ext}_{C(R)}^{i+1}(X,K_1)$. 以此类推，通过维数转移可知 $\mathrm{Ext}_{C(R)}^i(X,Y)\cong\mathrm{Ext}_{C(R)}^{n+i}(X,W_n)=0$.

定理 1 的证明　考虑复形的 $\widetilde{\mathrm{W}}$ 分解 $\cdots\to W_1\to W_0\to X\to 0$. 令 $K_1=\mathrm{Ker}(W_0\to X)$, $K_{i+1}=\mathrm{Ker}(K_i\to K_{i-1})$ $(\forall i\geqslant 1)$. 任意 W 复形是 W-Gorenstein 复形，故由 $\widetilde{\mathrm{WG}}$-resdim$(X)=n$ 可知 $K_n\in\widetilde{\mathrm{WG}}$.

由 W-Gorenstein 复形的定义，存在正合序列 $0\to K_n\to V^0\to V^1\to\cdots$，其中 $V^i\in\widetilde{\mathrm{W}}$，且对任意 $W\in\widetilde{\mathrm{W}}$，用 $\mathrm{Hom}_{C(R)}(-,W)$ 和 $\mathrm{Hom}_{C(R)}(W,-)$ 作用在该序列上得到的序列仍然是正合的. 令

$$L^1=\mathrm{Coker}(G\to V^0),\quad L^{i+1}=\mathrm{Coker}(V^{i-1}\to V^i)\quad(\forall i\geqslant 1)$$

从而，由正合序列 $\mathrm{Hom}_{C(R)}(V^0,W_{n-1})\to\mathrm{Hom}_{C(R)}(K_n,W_{n-1})\to 0$ 可知存在态射 $f_0:V^0\to$

W_{n-1} 及 $g_0:L^1\to K_{n-1}$，使得图 2 是可交换的.

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r] \arrow[d, equal] & V^0 \arrow[r] \arrow[d, "f_0"] & L^1 \arrow[r] \arrow[d, "g_0"] & 0 \\
0 \arrow[r] & K_n \arrow[r] & W_{n-1} \arrow[r] & K_{n-1} \arrow[r] & 0
\end{tikzcd}$$

图 2　交换图

Fig. 2　Commutative diagram

同理，存在态射 $f_1:V^1\to W_{n-2}$ 及 $g_1:L^2\to K_{n-2}$，使得图 3 是可交换的.

$$\begin{tikzcd}
0 \arrow[r] & L^1 \arrow[r] \arrow[d, "g_0"] & V^1 \arrow[r] \arrow[d, "f_1"] & L^2 \arrow[r] \arrow[d, "g_1"] & 0 \\
0 \arrow[r] & K_{n-1} \arrow[r] & W_{n-2} \arrow[r] & K_{n-2} \arrow[r] & 0
\end{tikzcd}$$

图 3　交换图

Fig. 3　Commutative diagram

以此类推，得到图 4 的交换图.

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r] \arrow[d, equal] & V^0 \arrow[r] \arrow[d, "f_0"] & V^1 \arrow[r] \arrow[d, "f_1"] & \cdots \arrow[r] & V^{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & L^n \arrow[r] \arrow[d, "f_n"] & 0 \\
0 \arrow[r] & K_n \arrow[r] & W_{n-1} \arrow[r] & W_{n-2} \arrow[r] & \cdots \arrow[r] & W_0 \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

图 4　交换图

Fig. 4　Commutative diagram

由图 4 可以得到一个正合序列，称之为映射锥：

$\mathbf{C}=0\to K_n\to K_n\oplus V^0\to W_{n-1}\oplus V^1\to\cdots\to W_1\oplus V^{n-1}\to W_0\oplus L^n\to X\to 0$.

同时，有正合序列 $\mathbf{K}=0\to K_n\to K_n\to 0\to\cdots$ 从而可以得到复形的正合序列

$\mathbf{C}/\mathbf{K}=0\to V^0\to W_{n-1}\oplus V^1\to\cdots\to W_1\oplus V^{n-1}\to W_0\oplus L^n\to X\to 0$

令 $G=W_0\oplus L^n$，$K=\mathrm{Ker}(W_0\oplus L^n\to X)$. 显然，$\widetilde{W}$-resdim$(K)=n-1$. 对于任意复形 $G'\in\widetilde{WG}$，由引理 1 可知 $\mathrm{Ext}^1_{C(R)}(G',K)=0$，从而 $f:G\to X$ 是 W-Gorenstein 预覆盖. 由于 K 是正合复形，故 f 是满的拟同构.

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教学研究

加权马尔可夫链在陇南地区降水量预测中的应用

王胜青

(陇南师范高等专科学校 数信学院, 甘肃 陇南 742500)

摘 要: 为了更好地指导农业生产, 应用马尔可夫链理论和方法, 依据陇南地区 1940–2015 年年降水量数据, 采用均值 - 标准差分级法, 并以规范化的各阶自相关系数为权, 建立加权马尔可夫链预测模型, 对陇南地区年降水量所处状态进行预测, 在此基础上采用状态特征值结合线性插值的方法, 对年降水量的具体数值予以推断, 获得了较为满意的结果.

关键词: 加权马尔可夫链; 转移矩阵; 年降水量; 预测

降水是大气环流和水文循环中的重要因子, 由于气象条件的多样性、变异性和复杂性, 降水过程存在大量的随机性, 准确地进行降水量预测, 既是制定水资源开发利用策略的科学依据, 又是防灾、抗灾、救灾的有效指导. 近年来, 随着马尔可夫理论的不断完善, 利用马尔可夫链模型进行降水量预测越来越受到众多学者们的关注. 冯耀龙, 等 [1] 针对河川径流量为相依随机变量的特点, 提出了以规范化的各阶自相关系数为权, 用加权的马尔可夫链来预测河流未来丰枯状况的方法. 但预测结果为河川径流量的状态, 而不是具体数值; 孙才志, 等 [2] 应用有序聚类的方法建立降水丰枯状况的分级标准, 然后针对降水量为相依随机变量的特点, 采取以规范化的各阶自相关系数为权重, 用加权的马尔可夫链模型来预测未来降水的丰枯变化状况. 同样预测结果为降水量的某一个状态, 而不是具体数值; 夏乐天 [3] 基于对统计试验的各种马尔可夫链预测方法的比较分析以及研究各种因素对马尔可夫链预测精度的影响, 得出加权马尔可夫链预测方法的预测精度最高、叠加马尔可夫链预测方法的预测精度次之、基于绝对分布的马尔可夫链预测方法的预测精度最低的结论; 袁建辉, 等 [4] 运用加权马尔可夫模型方法预测年降水量的降水状态后, 采用向量叠加法来预测年降水量的具体数值. 本文利用陇南水文局 1940 -2015 年陇南市年降水量观测数据, 采用均值 - 标准差分级法, 建立了适用于陇南地区年降水量的分级标准, 并以规范化的各阶自相关系数为权, 建立加权马尔可夫链预测模型, 对陇南地区 2016 年的年降水量所处状态进行预测, 在此基础上采用状态特征值结合线性插值的方法, 对 2016 年的年降水量具体数值进行预测.

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1 马尔可夫链和预测模型的构建

1.1 马尔可夫链的定义 [5]

设$\{X_n,\ n\geq 0\}$为随机过程, I 为其状态空间, 若对任意的整数 $n\geq 0$ 和任意的 $i_0,i_1,\cdots,i_{n+1}\in I$, 条件概率满足 $P\{X_{n+1}=i_{n+1}|X_0=i_0,\ X_1=i_1,\cdots,X_n=i_n\}=P\{X_{n+1}=i_{n+1}|X_n=i_n\}$, 则称$\{X_n,\ n\geq 0\}$为马尔可夫链.

1.2 转移概率矩阵 [5]

$\forall i,j\in I$, 称 $p_{ij}(n)=P\{X_{n+1}=j\,|X_n=i\}$ 为马尔可夫链 $\{X_n,\ n\geq 0\}$在 n 时刻的一步转移概率. 当转移概率 $p_{ij}(n)$ 与时刻 n 无关时, 称此马尔可夫链为齐次的.

用 P 表示一步转移概率 p_{ij} 所组成的矩阵, 即

$$P=\begin{pmatrix} p_{11} & p_{12}\cdots & p_{1j}\cdots \\ p_{21} & p_{22}\cdots & p_{2j}\cdots \\ \cdots & \cdots\cdots & \cdots\cdots \\ p_{i1} & p_{i2}\cdots & p_{ij}\cdots \\ \cdots\cdots & \cdots & \cdots\cdots \end{pmatrix}$$

称 P 为一步转移概率矩阵.

$\forall i,j\in I,\quad k\geq 1, p_{ij}^{(k)}=P\{X_{m+k}=j|X_m=i\}$为马尔可夫链$\{X_n,\ n\geq 0\}$的 k 步转移概率, 并称 $P^{(k)}=(p_{ij}^{(k)})$ 为马尔可夫链的 k 步转移矩阵.

不难验证其满足以下两个性质:

1) $p_{ij}^{(k)}\geq 0$;

2) $\sum\limits_{j\in I}p_{ij}^{(k)}=1$.

当 $k=1$ 时, $p_{ij}^{(1)}=p_{ij}$, $P^{(1)}$ 即为一步转移概率矩阵 P.

1.3 加权马尔可夫预测模型的求解步骤

1) 计算序列均值 $\overline{x}$ 和均方差 s, 建立分级标准 (相当于确定马尔可夫链的状态空间).

2) 按 "1)" 所建立的分级标准, 确定序列中各个时段所处的状态.

3) 对年降水量状态序列进行马氏性检验.

4) 计算降水序列各阶自相关系数 r_k,

$$r_k=\sum_{i=1}^{n-k}(x_i-\overline{x})(x_{i+k}-\overline{x})/\sum_{i=1}^{n}(x_i-\overline{x})^2$$

式中：r_k— 第 k 阶 (滞时为 k) 自相关系数; x_i— 第 i 时段的年降水量; $\overline{x}$— 多年平均值; n—时间长度.

5) 对各阶自相关系数规范化. $w_k=|r_k|/\sum\limits_{k=1}^{m}|r_k|$, 式中：$m$— 预测时需要计算到的最大阶数, m 取值为 5.

6) 对 2) 式所得到结果进行统计, 得到不同步长的马氏转移矩阵, 决定了年降水量状态转移过程中的概率法则.

7) 分别以前面各自的年降水量为初始状态, 结合其相应的状态转移矩阵即可预测出该时段降水量的状态概率 $p_i^{(k)}$, 其中 i 为状态, k 为步长 $(k=1,2,\cdots,m)$.

8) 将同一状态的各预测概率加权和作为年降水量处于该状态的预测概率, 即 $p_i = \sum\limits_{i=1}^{m} w_i p_i^{(k)}$, 则 $\max\{p_i, i \in I\}$(I 为其状态空间) 所对应的 i 即为该时段降水量的预测状态. 再重复以上步骤, 可进行下时段指标值状态的预测.

9) 要预测指标值的具体数值, 可采用状态特征值结合线性插值的方法予以推断 [3]. 具体推断如下：先算得状态特征值 $\mu = \sum\limits_{j\in I} j^{\beta} p_j$, 其中 $\beta > 0$ 为调整因子. 正常情况下 $i \le \mu < i+1$, 设对应的状态 i 的区间下限为 a, 上限为 b, 则要预测的指标值 x 可用下限法计算得 $x = a + (-1)^k(\mu - i)(b - a)$.

也可用上限法计算得 $x = b - (-1)^k(i + 1 - \mu)(b - a)$, 其中

$$k = \begin{cases} 0, \text{状态值增加与指标值增加一致} \\ 1, \text{否则} \end{cases}.$$

10) 可进一步对该马尔可夫链的特征 (遍历性、平稳分布等) 进行分析.

2 加权马尔可夫链在陇南地区降水量预测中的应用

2.1 数据来源及降水量分级标准的确定

对于降水量的分级, 常用方法是采用均值 - 标准差分级法 [1]. 设降水量序列为 $X_1, X_2, \cdots, X_n$, 样本均值为 $\overline{x}$, 样本标准差为 $s = \sqrt{\frac{1}{n-1}\sum\limits_{i=1}^{n}(x_i - \overline{x})^2}$, 如果这是一个弱相关序列 (相关系数的绝对值小于或等于 0.2), 则可近似看作是独立同分布的序列. 于是, 可按年降水量指标值是否落在 $(-\infty, \overline{x}-1.1s)$、$(\overline{x}-1.1s, \overline{x}-0.5s)$、$(\overline{x}-0.5s, \overline{x}+0.5s)$、$(\overline{x}+0.5s, \overline{x}+1.1s)$、$(\overline{x}+1.1s, +\infty)$ 内, 将各年降水量分为枯水年、偏枯年、平水年、偏丰年和丰水年 5 个状态 (相当于确定马尔可夫链的状态空间).

根据陇南市水文局统计数据 (表 2), 用 Eviews 数据分析软件绘制其时序图, 为平稳时间序列 (图 1).

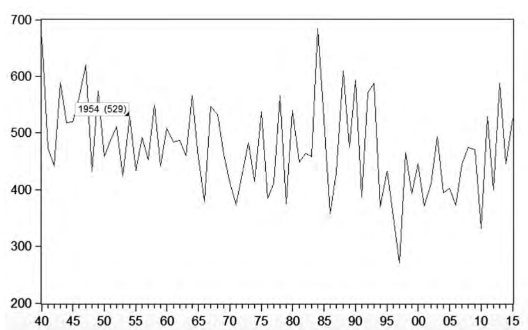


图 1 降水量时序图

陇南市 1940–2015 年降水序列的均值及均方差无偏估计值由下式计算得出：

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = 473.1\text{mm}, \quad s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2} = 80.4\text{mm}$$

显然, 样本值接近均值 $\overline{x}$ 时正常, 越小于 $\overline{x}$ 越旱, 越大于 $\overline{x}$ 越涝. 记枯水年、偏枯年、平水年、偏丰年和丰水年 5 个状态为 I={1, 2, 3, 4, 5}, 各状态对应的降水量区间见表 1.

依据表 1 中的分级标准对陇南地区 1940−2015 年降水量进行分级, 结果如表 2 所示.

2.2 进行马氏检验

检验随机过程是否符合马尔可夫性质是应用马尔可夫模型的必要前提. 通常离散序列的马氏链可用 χ^2 统计量来检验.

表 1 年降水量分级表

状态	级别	分级标准	降水量区间/mm
1	枯水年	$x < \overline{x} - 1.1s$	$x < 384.7$
2	偏枯年	$\overline{x} - 1.1s \le x < \overline{x} - 0.5s$	$384.7 \le x < 432.9$
3	平水年	$\overline{x} - 0.5s \le x < \overline{x} + 0.5s$	$432.9 \le x < 513.3$
4	偏丰年	$\overline{x} + 0.5s \le x < \overline{x} + 1.1s$	$513.3 \le x < 561.5$
5	丰水年	$x \ge \overline{x} + 1.1s$	$x \ge 561.5$

表 2 陇南市年平均降水量

年份	降水量 (mm)	状态等级	年份	降水量 (mm)	状态等级	年份	降水量 (mm)	状态等级
1940	668.5	5	1966	379.7	1	1992	570.7	5
1941	473.6	3	1967	547.0	4	1993	587.8	5
1942	441.6	3	1968	532.9	4	1994	369.1	1
1943	588.4	5	1969	462.0	3	1995	433.3	3
1944	517.6	4	1970	411.9	2	1996	351.8	1
1945	519.1	4	1971	373.9	1	1997	270.1	1
1946	560.9	4	1972	428.8	2	1998	464.4	3
1947	620.7	5	1973	482.1	3	1999	392.6	2
1948	432.4	2	1974	415.1	2	2000	445.8	3
1949	574.2	5	1975	536.7	4	2001	370.3	1
1950	458.5	3	1976	385.2	2	2002	410.6	2
1951	487.4	3	1977	412.2	2	2003	493.3	3
1952	511.9	3	1978	566.3	5	2004	394.7	2
1953	423.5	2	1979	375.4	1	2005	402.3	2
1954	528.7	4	1980	540.0	4	2006	372.0	1
1955	433.7	3	1981	448.7	3	2007	443.9	3
1956	492.3	3	1982	463.6	3	2008	474.9	3
1957	452.4	3	1983	457.8	3	2009	470.7	3
1958	549.5	4	1984	684.5	5	2010	331.9	1
1959	441.3	3	1985	523.8	4	2011	529.0	4
1960	508.5	3	1986	357.3	1	2012	399.1	2
1961	484.1	3	1987	429.7	2	2013	588.1	5
1962	487.2	3	1988	609.8	5	2014	446.0	3
1963	459.6	3	1989	474.1	3	2015	526.5	4
1964	565.2	5	1990	593.0	5			
1965	456.4	3	1991	387.9	2			

由概率论知识可知, 统计量 $\chi^2 = 2\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{m} f_{ij}\left|\ln\frac{p_{ij}}{p_{\bullet j}}\right|$ 服从自由度为 $(m-1)^2$ 的 χ^2 分布,

其中 f_{ij} 为降水量序列中从 i 状态一步转移到 j 状态的转移频率, p_{ij} 为降水量序列中从 i 状态一步转移到 j 状态的转移概率, $p_{\bullet j}=\sum\limits_{i=1}^{m} f_{ij}/\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{m} f_{ij}$ 为边际概率. 由表 2 提供的数据可算得:

$$(f_{ij})_{5\times5}=\begin{pmatrix}1&3&3&3&0\\2&2&3&2&5\\4&5&13&2&4\\1&2&4&3&1\\2&2&5&2&1\end{pmatrix},\quad (p_{ij})_{5\times5}=\begin{pmatrix}1/10&3/10&3/10&3/10&0\\2/14&2/14&3/14&2/14&5/14\\4/28&5/28&13/28&2/28&4/28\\1/11&2/11&4/11&3/11&1/11\\2/12&2/12&5/12&2/12&1/12\end{pmatrix}.$$

表 3　边际概率表

状态	1	2	3	4	5
$p_{\bullet j}$	10/75	14/75	28/75	11/75	12/75

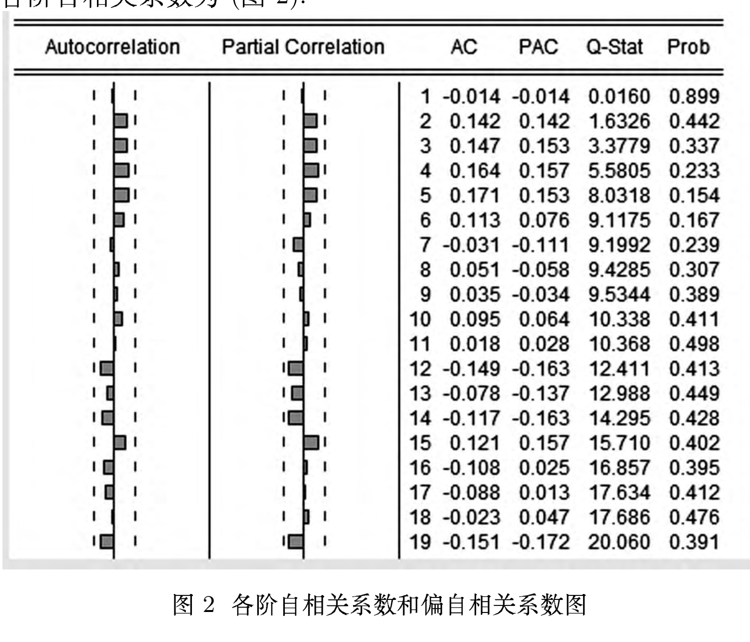
表 4　统计量 $\chi^2=2\sum\limits_{i=1}^{5}\sum\limits_{j=1}^{5} f_{ij}\left|\ln\frac{p_{ij}}{p_{\bullet j}}\right|$ 计算表

状态	$f_{i1}\left\|\ln\frac{p_{i1}}{p_{\bullet 1}}\right\|$	$f_{i2}\left\|\ln\frac{p_{i2}}{p_{\bullet 2}}\right\|$	$f_{i3}\left\|\ln\frac{p_{i3}}{p_{\bullet 3}}\right\|$	$f_{i4}\left\|\ln\frac{p_{i4}}{p_{\bullet 4}}\right\|$	$f_{i5}\left\|\ln\frac{p_{i5}}{p_{\bullet 5}}\right\|$	合计
1	0.2877	1.4234	0.6561	2.1469	0	4.5141
2	0.1380	0.5350	1.6655	0.0526	4.0148	6.4059
3	0.2760	0.2217	2.8344	1.4389	0.4533	5.2243
4	0.3830	0.0526	0.1053	1.8609	0.5653	2.9671
5	0.4463	0.2267	0.5491	0.2557	0.6523	2.1301
						21.2415

计算得 $\chi^2=2\times21.2415=42.483$. 给定显著性水平 α=0.05, 查表得: $\chi^2_\alpha(16)=26.296$, 由于 $\chi^2>\chi^2_\alpha((m-1)^2)$, 故年降水量序列满足马氏性.

2.3　计算降水序列各阶自相关系数

由公式: $r_k=\sum\limits_{i=1}^{n-k}(x_i-\overline{x})(x_{i+k}-\overline{x})/\sum\limits_{i=1}^{n}(x_i-\overline{x})^2, k=1,2,3,4,5$ 或利用 Eviews 数据分析软件计算得各阶自相关系数为 (图 2).



Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	-0.014	-0.014	0.0160	0.899
		2	0.142	0.142	1.6326	0.442
		3	0.147	0.153	3.3779	0.337
		4	0.164	0.157	5.5805	0.233
		5	0.171	0.153	8.0318	0.154
		6	0.113	0.076	9.1175	0.167
		7	-0.031	-0.111	9.1992	0.239
		8	0.051	-0.058	9.4285	0.307
		9	0.035	-0.034	9.5344	0.389
		10	0.095	0.064	10.338	0.411
		11	0.018	0.028	10.368	0.498
		12	-0.149	-0.163	12.411	0.413
		13	-0.078	-0.137	12.988	0.449
		14	-0.117	-0.163	14.295	0.428
		15	0.121	0.157	15.710	0.402
		16	-0.108	0.025	16.857	0.395
		17	-0.088	0.013	17.634	0.412
		18	-0.023	0.047	17.686	0.476
		19	-0.151	-0.172	20.060	0.391

图 2　各阶自相关系数和偏自相关系数图

$$r_1=-0.014, r_2=0.142, r_3=0.147, r_4=0.164, r_5=0.171.$$

利用公式：$w_k = |r_k| / \sum_{k=1}^{m} |r_k|$, 对各阶自相关系数规范化得到不同滞时 (步长) 的权重 $w_1=0.022, w_2=0.223, w_3=0.230, w_4=0.257, w_5=0.268$.

2.4 计算不同步长的马氏转移矩阵

根据年降水量所处的状态, 利用 MATLAB 软件计算不同步长的马尔可夫链的状态概率转移矩阵：

$$P^{(1)}=\begin{pmatrix} 0.1000 & 0.3000 & 0.3000 & 0.3000 & 0 \\ 0.1429 & 0.1429 & 0.2143 & 0.1429 & 0.3571 \\ 0.1429 & 0.1786 & 0.4643 & 0.0714 & 0.1429 \\ 0.0909 & 0.1818 & 0.3636 & 0.2727 & 0.0909 \\ 0.1667 & 0.1667 & 0.4167 & 0.1667 & 0.0833 \end{pmatrix} \quad P^{(2)}=\begin{pmatrix} 0.1230 & 0.1810 & 0.3427 & 0.1761 & 0.1773 \\ 0.1378 & 0.1870 & 0.3737 & 0.1771 & 0.1244 \\ 0.1364 & 0.1881 & 0.3822 & 0.1448 & 0.1485 \\ 0.1270 & 0.1829 & 0.3721 & 0.1688 & 0.1493 \\ 0.1290 & 0.1924 & 0.3745 & 0.1629 & 0.1411 \end{pmatrix}$$

$$P^{(3)}=\begin{pmatrix} 0.1327 & 0.1855 & 0.3727 & 0.1648 & 0.1444 \\ 0.1307 & 0.1877 & 0.3711 & 0.1638 & 0.1466 \\ 0.1330 & 0.1871 & 0.3732 & 0.1593 & 0.1473 \\ 0.1322 & 0.1862 & 0.3736 & 0.1617 & 0.1463 \\ 0.1322 & 0.1862 & 0.3719 & 0.1609 & 0.1488 \end{pmatrix} \quad P^{(4)}=\begin{pmatrix} 0.1320 & 0.1869 & 0.3727 & 0.1619 & 0.1465 \\ 0.1322 & 0.1865 & 0.3724 & 0.1616 & 0.1472 \\ 0.1324 & 0.1868 & 0.3726 & 0.1613 & 0.1469 \\ 0.1323 & 0.1868 & 0.3728 & 0.1614 & 0.1468 \\ 0.1324 & 0.1867 & 0.3727 & 0.1615 & 0.1467 \end{pmatrix}$$

$$P^{(5)}=\begin{pmatrix} 0.1323 & 0.1867 & 0.3726 & 0.1615 & 0.1469 \\ 0.1323 & 0.1867 & 0.3727 & 0.1615 & 0.1468 \\ 0.1323 & 0.1868 & 0.3726 & 0.1615 & 0.1469 \\ 0.1323 & 0.1867 & 0.3726 & 0.1615 & 0.1469 \\ 0.1323 & 0.1868 & 0.3726 & 0.1615 & 0.1468 \end{pmatrix}$$

2.5 马氏链预测模型前期检验

验证 2014 年降水量状态及数值. 根据陇南市 2009-2013 年五年的观测降水量数值及其相应的状态转移概率矩阵, 结合上面计算所得的马尔可夫链的前 5 阶权重值 $w_k(k=1,2,3,4,5)$, 对 2014 年降水量进行预测验证, 其结果见表 5.

表 5 2014 年降水状态预测结果

初始年	状态	滞时	权重	2014 年各降水状态概率				
				1	2	3	4	5
2013	5	1	0.022	0.1667	0.1667	0.4167	0.1667	0.0833
2012	2	2	0.223	0.1378	0.1870	0.3737	0.1771	0.1244
2011	4	3	0.230	0.1322	0.1862	0.3736	0.1617	0.1463
2010	1	4	0.257	0.1320	0.1869	0.3727	0.1619	0.1465
2009	3	5	0.268	0.1323	0.1868	0.3726	0.1615	0.1469
$p_i=\sum_{i=1}^{5} w_i p_i^{(k)}$				0.1342	0.1863	0.3741	0.1652	0.1402

由上表知 $\max(p_i)=0.3741$, 此时 $i=3$. 即 2014 年降水量预测状态为 3, 而 2014 年降水量实际状态也为 3, 和预测状态相吻合.

进一步采用状态特征值结合线性插值的方法予以推断. 对于调整因子 $\beta=1.03$, 由公式

$\mu=\sum\limits_{j\in I} j^{\beta}p_j$ 求得状态特征值为 $3.0991(i\le\mu<i+1)$. 用下限法计算公式求得

$$x=a+(-1)^k(\mu-i)(b-a)=432.9+0.0991(80.4)=440.87\text{mm}(k=0)$$

此值与实际值 446mm 的相对误差仅为 1.15%, 与实际值基本吻合.

验证 2015 年降水量状态及数值. 根据陇南市 2010–2014 年五年的观测降水量数值及其相应的状态转移概率矩阵, 运用计算所得的马尔可夫链的前 5 阶权重值 $w_k(k=1,2,3,4,5)$, 对 2015 年降水量进行预测验证, 其结果见表 6.

表 6　2015 年降水状态预测结果

初始年	状态	滞时	权重	2015 年各降水状态概率				
				1	2	3	4	5
2014	3	1	0.022	0.1429	0.1786	0.4643	0.0714	0.1429
2013	5	2	0.223	0.1290	0.1924	0.3745	0.1629	0.1411
2012	2	3	0.230	0.1307	0.1877	0.3711	0.1638	0.1466
2011	4	4	0.257	0.1323	0.1868	0.3728	0.1614	0.1468
2010	1	5	0.268	0.1323	0.1867	0.3726	0.1615	0.1469
$p_i=\sum\limits_{i=1}^{5} w_i p_i^{(k)}$				0.1314	0.1880	0.3747	0.1603	0.1454

由表 6 知 $\max(p_i)=0.3747$, 此时 $i=3$. 即 2015 年降水量预测状态为 3, 而 2015 年降水量实际状态为 4.

进一步采用状态特征值结合线性插值的方法, 对于调整因子 $\beta=1.03$, 由公式 $\mu=\sum\limits_{j\in I} j^{\beta}p_j$ 求得状态特征值为 $3.1085(i\le\mu<i+1)$. 用下限法计算公式求得 $x=a+(-1)^k(\mu-i)(b-a)=432.9+0.1085(80.4)=441.62\text{mm}$.

由此可见, 虽然降水状态不符合, 但预测值 441.62mm 与实际值 526.5mm 的相对误差为 16.12%, 与实际值较吻合 (一般认为相对误差 $<20\%$ 就是令人满意的预测结果), 说明该方法在预测中长期降水量中是可行的.

2.6　预测陇南地区未来年降水量

用 2011–2015 年五年的降水量数值及其相应的状态转移概率矩阵, 结合上面计算所得的马尔可夫链的前 5 阶权重值 $w_k(k=1,2,3,4,5)$, 对陇南地区 2016 年降水量进行预测, 其结果见表 7.

表 7　2016 年降水状态预测结果

初始年	状态	滞时	权重	2016 年各降水状态概率				
				1	2	3	4	5
2015	4	1	0.022	0.0909	0.1818	0.3636	0.2727	0.0909
2014	3	2	0.223	0.1364	0.1881	0.3822	0.1448	0.1485
2013	5	3	0.230	0.1322	0.1862	0.3719	0.1609	0.1488
2012	2	4	0.257	0.1322	0.1865	0.3724	0.1616	0.1472
2011	4	5	0.268	0.1323	0.1867	0.3726	0.1615	0.1469
$p_i=\sum\limits_{i=1}^{5} w_i p_i^{(k)}$				0.1323	0.1867	0.3743	0.1601	0.1465

由上表知 $\max(p_i)=0.3743$, 此时 $i=3$. 即 2016 年降水量预测状态为 3.

对于调整因子 $\beta = 1.03$, 由公式 $\mu = \sum\limits_{j\in I} j^{\beta} p_j$ 求得状态特征值为 3.1108. 用下限法计算公式求得 $x = \underline{x} + (-1)^k(\mu - i)(\overline{x} - \underline{x}) = 432.9 + 0.1108(80.4) = 441.8\text{mm}$.

即陇南地区 2016 年降水量预测值为 441.8mm, 仍属于平水年.

3 结论

本文运用陇南地区 1940–2015 年降水量资料, 采用均值 - 标准差分级法对降水量数据序列进行分级, 并以规范化的各阶自相关系数为权, 建立了加权马尔可夫链预测模型, 通过验证该模型在降水量所处状态的预测中准确率较高. 在此基础上进一步采用状态特征值结合线性插值的方法, 对年降水量具体数值进行预测, 相对误差在人们普遍认可的范围内. 通过预测, 陇南地区降水量在未来年内属于平水年, 降水正常, 河流水量与多年平均情况相当, 可以大面积种植农作物, 农业丰收的概率较大.

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Prediction of Precipitation Based on the Weighted Markov Chain in Longnan Area

WANG Sheng-qing

(College of Mathematics and Information Technology, Longnan Teachers College, Longnan 742500, China)

Abstract: In order to guide agricultural production better, this paper applies the theory and methods of the Markov chain, according to the Longnan area annual rainfall datas from 1940 to 2015, the mean standard deviation method is used to standardize the slf correlation coefficients as weighs, a weighted Markov chain prediction model is established to forecast the annual rainfall of Longnan area. On this basis, it uses the eigenvalue and linear interpolation to analyze the concrete value of annual rainfall, a satisfactory result has been got.

Keywords: weighted Markov chain; transfer matrix; annual precipitation; prediction

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实数序列的通项公式问题

东洪平
（陇南师范高等专科学校 数学系，甘肃 成县　742500）

摘要：讨论实数序列的通项公式，证明了任意实数序列可以通过$[0,+\infty)$上任意次连续可微函数给出其通项公式，任意单调有界实数序列可以通过$(-\infty,+\infty)$上的解析函数给出其通项公式.

关键词：实数序列；单调有界数列；通项公式
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On the formulas of general terms of real sequences

DONG Hong-ping
（Department of Mathematics，Longnan Teachers College，Chengxian 742500，Gansu，China）

Abstract：This paper is concerned with the formulas of general terms of sequences of real numbers. It is proved that for any sequence of real numbers，the formula of general terms of the sequence can be given via continuously differentiable functions on $[0,+\infty)$，and for any bounded monotone sequence of real numbers，its general term formula can be obtained by using analytic functions on $(-\infty,+\infty)$.

Key words：sequence of real numbers；monotonically bounded sequence；formulas of general terms

0　引言

在中学数学中，常常给定一个数列所满足的关系式，要求写出该数列的通项公式. 受这类问题的启发，本文提出一个更加一般性的问题：

问题 1　对任意实数序列 $l=\{a_n\}_{n=1}^{N}(N\leqslant+\infty)$，能否给出 l 的通项公式？

要回答这个问题，我们首先需要澄清一些基本概念.

定义 1　一个数列 l 是指定义在一个可数有序集 S 上的一个实值函数 $f(n)(n\in S)$，很多情况下这个有序集合就取自然数集的一个子集[1-2].

不失一般性，不妨取

$$S=\{n\in\mathrm{N}:1\leqslant n\leqslant N\},$$

此处 $N\leqslant+\infty$. 若 $N<+\infty$，则称数列 l 为有穷数列，否则称为无穷数列. 为方便起见，常常记 $a_n=f(n)$，从而可将一个数列 l 写成常见形式 $l=\{a_n\}_{n=1}^{N}(N\leqslant+\infty)$. 如此一来，上述问题就变得比较明确了，给出所谓通项公式，其实就是要明确函数关系 f. 于是一个等价的提法是：已知函数 f 的所有函数值

$$f(n)=a_n,\ 1\leqslant n\leqslant N\leqslant+\infty,\qquad(1)$$

能否给出函数 f 的表达式？这样一来问题似乎又显得有点平凡，因为理论上(1)式本身就可以看成是函数 f 的一个表达式，但是如果将函数 f 限定在某个特定的函数类中，问题就有点不平凡了. 事实上，回答这个看似简单的问题，其难度远远超出我们的预料. 本文将紧紧围绕这一问题进行深入探讨.

1　主要结果

在一个特定的函数类中寻找给定数列的通项公式，其更为严格的数学描述如下：

问题 2　任给数列 $l=\{a_n\}_{n=1}^{N}(N\leqslant+\infty)$和函

收稿日期：2014-12-06；修改稿收到日期：2015-06-17
作者简介：东洪平（1963—），男，甘肃成县人，副教授. 主要研究方向为数学分析.
E-mail：donghongping@163.com

数族 F，是否存在 $f\in$ F 使得（1）式成立？

解答问题 2 的关键在于函数族 F 的选取.

1.1　连续函数族和连续可微函数族的情形

先在连续函数族中考虑这一问题，此时可取 F $=C[0,+\infty)$，我们有

定理 1　对任一数列 $l=\{a_n\}_{n=1}^N(N\leqslant+\infty)$，存在 $f\in$ F，使得 $a_n=f(n)$.

证明　只考虑 $N=+\infty$，即 l 是无穷数列的情形. 为方便起见，补充 $a_0=0$，定义函数

$$f(x)=\begin{cases}a_n,\ x\in\left(n-\dfrac{1}{4},n+\dfrac{1}{4}\right),\\ a_n+2(a_{n+1}-a_n)\left[x-\left(n+\dfrac{1}{4}\right)\right],\\ \qquad x\in\left[n+\dfrac{1}{4},n+\dfrac{3}{4}\right],\end{cases}\tag{2}$$

$n=0,1,2,\cdots$，则 $f(x)$ 即为所求.

事实上，对每个 n，f 在开区间 $\left(n-\dfrac{1}{4},n+\dfrac{1}{4}\right)$ 和闭区间 $\left[n+\dfrac{1}{4},n+\dfrac{3}{4}\right]$ 上分别连续. 在端点 $x=n+\dfrac{1}{4}$ 处，由 f 的表达式有

$$f\left(n+\frac{1}{4}\right)=a_n=\lim_{x\to\left(n+\frac{1}{4}\right)^-}f(x);$$

在端点 $x=n+\dfrac{3}{4}$ 处，有

$$f\left(n+\frac{3}{4}\right)=a_n+2(a_{n+1}-a_n)\cdot\frac{1}{2}=a_{n+1}=\lim_{x\to\left(n+\frac{3}{4}\right)^+}f(x).$$

由此可知 f 在（2）式中区间的每一个端点处连续，从而 $f\in$ F. 易见 $f(n)=a_n(1\leqslant n<+\infty)$.　】

既然任何一个数列都可以通过连续函数给出通项公式，我们自然进一步要问：对任意数列 $l=\{a_n\}_{n=1}^N(N\leqslant+\infty)$，是否存在连续可微的实值函数 f 使得（1）式成立？回答是肯定的. 事实上，只要我们对（2）式中的函数 f 进行适当的磨光，就可以得到满足条件的光滑函数.

定理 2　对任意数列 $l=\{a_n\}_{n=1}^N(N\leqslant+\infty)$，存在 $[0,+\infty)$ 上任意次连续可微的函数 g，使得 $a_n=g(n)$.

证明　固定一个正数 ε 满足 $\varepsilon<\dfrac{1}{16}$. 令

$$\eta(x)=\begin{cases}\dfrac{1}{\alpha}e^{\frac{1}{|x|^2-\varepsilon^2}},\ |x|<\varepsilon,\\ 0,\ |x|\geqslant\varepsilon,\end{cases}$$

其中 $\alpha=\int_{-\varepsilon}^{\varepsilon}e^{\frac{1}{|x|^2-\varepsilon^2}}dx$. 容易验证 η 是 R 上任意次连续可微的函数，且

$$\int_{\infty}^{+\infty}\eta(x)dx=1,\ \eta(x)\equiv0(|x|\geqslant\varepsilon).$$

设 $f(x)$ 是由（2）式所定义的函数，对 $x<0$，补充定义 $f(x)=0$，则 $f(x)$ 是 $(-\infty,+\infty)$ 上的连续函数. 令

$$g(x)=\int_{-\infty}^{+\infty}\eta(x-y)f(y)dy,$$

我们证明 $g(x)$ 即为所求.

由 η 的定义，有 $\eta(x)=0(|x|\geqslant\varepsilon)$，从而对任意 $k\geqslant0$，有 $\eta^{(k)}(x)=0(|x|\geqslant\varepsilon)$，于是若 $|y-x|\geqslant\varepsilon$，则

$$\eta^{(k)}(x-y)=0,\ k=0,1,2,\cdots.\tag{3}$$

所以对任意有界区间 $I=[a,b]$，当 $x\in I$ 时，我们有

$$\int_{-\infty}^{+\infty}\frac{d^k}{dx^k}\eta(x-y)f(y)dy=\int_{a-\varepsilon}^{b+\varepsilon}\frac{d^k}{dx^k}\eta(x-y)f(y)dy.\tag{4}$$

（4）式左端作为含参量 x 的积分关于 $x\in I$ 一致收敛，由数学分析知识[3-4]可知，g 在 I 上任意次连续可微，再由 I 的任意性可知，g 是 $(-\infty,+\infty)$ 上任意次连续可微的函数.

若 $x\in\left[n-\dfrac{1}{16},n+\dfrac{1}{16}\right](n=1,2,\cdots)$，则由（4）式可知

$$g(x)=\int_{-\infty}^{+\infty}\eta(x-y)f(y)dy=\int_{n-\frac{1}{16}-\varepsilon}^{n+\frac{1}{16}+\varepsilon}\eta(x-y)f(y)dy.$$

注意到

$$\left[n-\frac{1}{16}-\varepsilon,n+\frac{1}{16}+\varepsilon\right]\subset\left[n-\frac{1}{8},n+\frac{1}{8}\right],$$

则由上式和函数 $f(x)$ 的定义，我们有

$$g(x)=\int_{n-\frac{1}{16}-\varepsilon}^{n+\frac{1}{16}+\varepsilon}\eta(x-y)a_n dy=a_n\int_{n-\frac{1}{16}-\varepsilon}^{n+\frac{1}{16}+\varepsilon}\eta(x-y)dy=a_n.$$

特别地，$g(n)=a_n$，所以 g 满足要求.　】

注 1：定理 2 的证明用到了偏微分方程理论中经典的"磨光技术"[5].

1.2　解析函数族的情形

由定理 1 和定理 2 可知，对任意数列 $l=\{a_n\}_{n=1}^N(N\leqslant+\infty)$，都存在一个定义在 $[0,+\infty)$ 上

的连续可微函数 $f(x)$，使得 $a_n=f(n)$. 然而我们更关心 $f(x)$ 是初等函数的情形，也就是说，是否可以用初等函数给出任意数列的通项公式？如果这个问题的答案是肯定的，那么通项公式的问题至少在理论上获得了一个满意的解答. 但非常遗憾的是，我们即举不出反例，也不能给出肯定的答复. 退一步看，一个有意思的问题是，若代之以初等函数类，我们是否可以用解析函数(即可以展开为幂级数或直接由幂级数所定义的函数)给出数列的通项公式？

若 $N<+\infty$，即 l 是有穷数列，则上述问题的回答是肯定的，此时令

$$f(x)=\sum_{m=1}^{N}a_m\left(\prod_{k\neq m}\frac{x-k}{m-k}\right),$$

则 $f(x)$ 为多项式，且

$$f(n)=\sum_{m=1}^{N}a_m\left(\prod_{k\neq m}\frac{n-k}{m-k}\right)=a_n,$$

从而 $f(x)$ 给出了数列的通项公式.

但当 $N=+\infty$ 时，问题要困难得多，我们至今没有一般性的结论. 不过对于单调数列，我们仍然获得了一个有意思的结果.

定义 2　数列 $l=\{a_n\}_{n=1}^{N}(N\leqslant+\infty)$ 称为单调递减的，若对任意的 n：$1\leqslant n\leqslant N$，均有 $a_n\geqslant a_{n+1}$.

类似可定义单调递增数列.

对任一数列 $l=\{a_n\}_{n=1}^{N}(N\leqslant+\infty)$(见定理 1)，由于存在 $[0,+\infty)$ 上的连续函数 $f(x)$ 使得 $f(n)=a_n$，因此其单调性可以通过连续函数给出定义，即有：

数列 l 称为单调递减的，若存在 $[0,+\infty)$ 上的连续函数 $f(x)$ 使得 $f(n)=a_n$，且

$$f(n)\geqslant f(n+1),\quad 1\leqslant n\leqslant N. \tag{5}$$

另一方面，若函数 $f(x)$ 满足条件(5)，则很容易构造 $[0,+\infty)$ 上一个新的单调递减的连续函数 $g(x)$ 使得 $g(n)=f(n)(1\leqslant n\leqslant N)$，因此数列 l 的单调性又可以直接用单调函数来定义.

定义 3　数列 l 称为单调递减的，若存在 $[0,+\infty)$ 上的单调递减的连续函数 $g(x)$，使得 $g(n)=a_n(1\leqslant n\leqslant N)$.

定理 3　设数列 $l=\{a_n\}_{n=1}^{N}(N\leqslant+\infty)$ 单调有界，则存在 $(-\infty,+\infty)$ 上的解析函数 $f(x)$，使得 $f(n)=a_n(n=1,2,\cdots)$.

证明　(i) 首先考虑数列 l 单调递减且 $\lim\limits_{n\to\infty}a_n=0$ 的情形.

我们从函数 $\sin\pi x$ 的幂级数表示

$$\sin\pi x=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(\pi x)^{2k-1}}{(2k-1)!} \tag{6}$$

入手. 由 Euler 对正弦函数的无穷乘积表示[6-7]，有

$$\sin\pi x=\pi x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right),$$

从而

$$\sum_{k=0}^{\infty}(-1)^k\frac{(\pi x)^{2k}}{(2k+1)!}=\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right). \tag{7}$$

上式右边的无穷乘积在 $[0,+\infty)$ 上是收敛的，且

$$\frac{\pi^{2k}}{(2k+1)!}=\sum_{1\leqslant j_1<\cdots<j_k}\frac{1}{j_1^2\cdots j_k^2}. \tag{8}$$

令

$$\varphi_n(x)=\frac{n\sin\pi x}{\pi x(x-n)},\quad n=1,2,\cdots, \tag{9}$$

则

$$\varphi_n(0)=\lim_{x\to0}\varphi_n(x)=-1,$$

$$\varphi_n(n)=\lim_{x\to n}\varphi_n(x)=(-1)^n.$$

此外，我们有

$$\varphi_n(m)=0,\quad\forall m\neq n.$$

将(6)式代入(9)式，有

$$\varphi_n(x)=-\left(1+\frac{x}{n}\right)\prod_{k=1,k\neq n}^{\infty}\left(1-\frac{x^2}{k^2}\right), \tag{10}$$

将(10)式右边的无穷乘积展开，可得

$$\varphi_n(x)=-\left(1+\frac{x}{n}\right)\sum_{k=0}^{\infty}(-1)^k c_{kn}x^{2k}, \tag{11}$$

其中

$$c_{0n}=1,\quad c_{kn}=\sum_{\substack{1\leqslant j_1<\cdots<j_k\\ j_i\neq n}}\frac{1}{j_1^2\cdots j_k^2},\quad k\geqslant1. \tag{12}$$

分别比较(7)和(10)式以及(8)和(12)式可知，(10)和(11)式右边在 $[0,+\infty)$ 上均收敛. 令

$$f_n(x)=(-1)^n a_n\varphi_n(x),$$

$$f(x)=\sum_{n=1}^{\infty}f_n(x), \tag{13}$$

则

$$\begin{aligned}f(x)=&\\ &\sum_{n=1}^{\infty}(-1)^n a_n\left[-\left(1+\frac{x}{n}\right)\right]\sum_{k=0}^{\infty}(-1)^k c_{kn}x^{2k}=\\ &-\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^{k+n}a_n c_{kn}x^{2k}-\\ &\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^{k+n}\frac{a_n}{n}c_{kn}x^{2k+1}.\end{aligned}$$

交换求和符号，可得

$$f(x)=-\sum_{k=0}^{\infty}(-1)^k\left[\sum_{n=1}^{\infty}(-1)^n a_n c_{kn}\right]x^{2k}-\sum_{k=0}^{\infty}(-1)^k\left[\sum_{n=1}^{\infty}(-1)^n\frac{a_n}{n}c_{kn}\right]x^{2k+1}=-\sum_{k=0}^{\infty}(-1)^{\frac{2k-1+(-1)^k}{4}}d_k x^k,$$

其中

$$d_{2k}=\sum_{n=1}^{\infty}(-1)^n a_n c_{kn},$$

$$d_{2k+1}=\sum_{n=1}^{\infty}(-1)^n\frac{a_n}{n}c_{kn}. \tag{14}$$

下面考虑(13)到(14)式中运算的合理性问题，为此只需说明其中级数的收敛性即可.

由于 $\varphi_n(0)=-1$，在 $x=0$ 处(13)式右边为一交错级数，由关于 a_n 的假设可知其收敛，从而 $f(0)$有定义. 设 $x>0$，则当 n 充分大时，$\varphi_n(x)$保持同号且单调有界，又 $\sum\limits_{n=1}^{\infty}(-1)^n a_n=-f(0)$ 收敛，由 Dirichlet 收敛准则[3]知(13)式右边级数收敛，故 $f(x)$定义合理.

对于(14)式，当 $k=0,1$ 时类似于检验(13)式右边在 $x=0$ 处的收敛性可知其中的级数收敛. 若 $k>1$，则由于 c_{kn} 关于 n 是单调有界的，从而级数 $\sum\limits_{n=1}^{\infty}(-1)^n c_{kn}$ 的部分和有界，又 a_n 与 $\dfrac{a_n}{n}$ 单调递减且趋于 0，所以由 Dirichlet 收敛准则可知(14)式中的级数收敛.

在每一个整数点 $x=n$ 处，有

$$f(n)=(-1)^n a_n\varphi_n(n)=a_n,$$

故 $f(x)$即为所求.

(ⅱ) 一般情形.

由 l 的有界性可知 $\lim\limits_{n\to\infty}a_n=a$ 存在. 先设 l 单调递减，此时令 $\alpha_n=a_n-a$，则数列$\{\alpha_n\}_{n=1}^{\infty}$单调递减且趋于 0. 由(ⅰ)中的结果可知存在$(-\infty,+\infty)$上的解析函数 $f(x)$，使得 $f(n)=\alpha_n(n=1,2,\cdots)$. 再令 $g(x)=f(x)+a$，则 $g(x)$是$(-\infty,+\infty)$上的解析函数，且 $g(n)=a_n(n=1,2,\cdots)$.

若 l 为单调递增数列，则令 $\beta_n=a-a_n$，易见数列$\{\beta_n\}_{n=1}^{\infty}$单调递减且趋于 0. 类似于刚才的讨论可得$(-\infty,+\infty)$上的解析函数 $g(x)$，满足 $g(n)=a_n(n=1,2,\cdots)$.　】

注 2：定理 3 的一个等价陈述是：设 $g(x)$是$[0,+\infty)$上的单调有界连续函数，则存在一个$(-\infty,+\infty)$上的解析函数 $f(x)$满足 $f(n)=g(n),n=1,2,\cdots$.

注 3：就中学数学而言，自然是期望能够在初等函数族 F 中给出问题 2 的一个回答. 对此，目前我们似乎还看不到希望.

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教学研究

广义 Cauchy 中值定理“中间点”的渐进性

杜争光

(陇南师范高等专科学校 数学系, 甘肃 成县 742500)

摘　要: 通过对广义 Cauchy 中值定理的讨论, 得到了广义 Cauchy 中值定理“中间点”渐进性的一个表达式, 并对已有的渐进性结果进行了推广.

关键词: Cauchy 中值定理; 中间点; 渐进性

1 引言及主要引理

近年来, 对于中值定理“中间点”渐进性的研究, 取得了一些进展, 并得到了一些重要结果 (见文献 [1-9]). 文献 [1] 对微分中值定理和积分中值定理进行了统一和推广, 并给出了广义 Cauchy 中值定理. 本文主要讨论广义 Cauchy 中值定理“中间点”的渐进性, 并对已有的“中间点”渐进性结果进行推广.

为叙述和讨论方便, 现将广义 Cauchy 中值定理引述如下:

引理 1[1] **(广义 Cauchy 中值定理)** 若函数 $f(x)$ 在闭区间 $[a,\ b]$ 上存在 $n+1$ 阶导数, 函数 $g(x)$ 在闭区间 $[a,b]$ 上存在 $m+1$ 阶导数, 且 $\forall x\in[a,b]$, $g^{(m+1)}(x)\neq 0$. 则至少存在一点 $\xi\in(a,b)$, 使得

$$\frac{f(b)-T_n[f(b)]}{g(b)-T_m[g(b)]}=\frac{m!f^{(n+1)}(\xi)}{n!g^{(m+1)}(\xi)}(b-\xi)^{n-m} \tag{1}$$

这里 $T_n[f(x)]=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$

引理 2 若函数 $f(x)$ 和 $g(x)$ 在 a 的某一邻域 $U(a)$ 内存在 $m+1$ 阶导数, 且存在 $\alpha\geq 0$ 和 $\beta\geq 0$, 对 $\forall x\in \overset{0}{U}(a)$ 有 $\lim\limits_{x\to a}\frac{f^{(m+1)}(x)}{(x-a)^\alpha}=A$ 和 $\lim\limits_{x\to a}\frac{g^{(m+1)}(x)}{(x-a)^\beta}=C$, 则有

$$\lim_{x\to a}\frac{f(x)-T_m[f(x)]}{(x-a)^{m+1+\alpha}}=\frac{A\Gamma(\alpha+1)}{\Gamma(m+2+\alpha)} \tag{2}$$

$$\lim_{x\to a}\frac{g(x)-T_m[g(x)]}{(x-a)^{m+1+\beta}}=\frac{C\Gamma(\beta+1)}{\Gamma(m+2+\beta)} \tag{3}$$

其中, $\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx$ 是 Γ 函数.

证明 由于函数 $f(x)$ 在 a 的邻域 $U(a)$ 内存在 m 阶导数, 且 $\lim\limits_{x\to a}\frac{f^{(m+1)}(x)}{(x-a)^\alpha}=A$, 利用 m 次 $L'Hospital$ 法则, 并注意到 Γ 函数的递推公式 $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, 就有

$$\lim_{x\to a}\frac{f(x)-T_m[f(x)]}{(x-a)^{m+1+\alpha}}$$

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$$
\begin{aligned}
&=\lim_{x\to a}\frac{f(x)-f(a)-\frac{f'(a)}{1!}(x-a)-\frac{f''(a)}{2!}(x-a)^2-\cdots-\frac{f^{(m)}(a)}{m!}(x-a)^m}{(x-a)^{m+1+\alpha}}\\
&=\lim_{x\to a}\frac{f'(x)-f'(a)-\frac{f''(a)}{1!}(x-a)-\cdots-\frac{f^{(m)}(a)}{(m-1)!}(x-a)^{m-1}}{(m+1+\alpha)(x-a)^{m+\alpha}}\\
&=\cdots\\
&=\lim_{x\to a}\frac{f^{(k)}(x)-f^{(k)}(a)-\frac{f^{(k+1)}(a)}{1!}(x-a)-\cdots-\frac{f^{(m)}(a)}{(m-k)!}(x-a)^{m-k}}{(m+1+\alpha)(m+\alpha)\cdots(m+\alpha-k)(x-a)^{m+1+\alpha-k}}\\
&=\cdots\\
&=\lim_{x\to a}\frac{f^{(m+1)}(x)}{(m+1+\alpha)(m+\alpha)\cdots(\alpha+1)(x-a)^{\alpha}}\\
&=\frac{A}{(m+1+\alpha)(m+\alpha-1)\cdots(\alpha+1)}\\
&=\frac{A\Gamma(\alpha+1)}{(m+1+\alpha)(m+\alpha-1)\cdots(\alpha+1)\Gamma(\alpha+1)}\\
&=\frac{A\Gamma(\alpha+1)}{\Gamma(m+2+\alpha)}
\end{aligned}
$$

即 (2) 式成立, 同理可证 (3) 式成立.

2 主要结果

定理 1 设函数 $f(x)$ 在闭区间 $[a,b]$ 上存在 $n+1$ 阶导数; 函数 $g(x)$ 在闭区间 $[a,b]$ 上存在 $m+1$ 阶导数, 且 $\forall x\in[a,b]$, $g^{(m+1)}(x)\neq 0$; 若存在 $\alpha\geq 0$ 和 $\beta\geq 0$, 对 $\forall x\in(a,b]$ 有 $\lim\limits_{x\to a}\frac{f^{(n+1)}(x)}{(x-a)^{\alpha}}=A$ 和 $\lim\limits_{x\to a}\frac{g^{(m+1)}(x)}{(x-a)^{\beta}}=C\neq 0$. 则由 (1) 式所确定的 “中间点”$\xi$ 满足

$$
\lim_{x\to a}\left(\frac{x-\xi}{x-a}\right)^{n-m}\left(\frac{\xi-a}{x-a}\right)^{\alpha-\beta}=\frac{\mathrm{B}(n+1,\alpha+1)}{\mathrm{B}(m+1,\beta+1)} \tag{4}
$$

其中, $B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx$ 是 B 函数.

证明 由于函数 $f(x)$ 在闭区间 $[a,b]$ 上存在 $n+1$ 阶导数, 函数 $g(x)$ 在闭区间 $[a,b]$ 上存在 $m+1$ 阶导数, 对 $\forall x\in(a,b]$, 构造辅助函数:

$$
F(x)=\frac{f(x)-T_n[f(x)]}{g(x)-T_m[g(x)]}(x-a)^{m+\beta-n-\alpha}
$$

注意到 $\lim\limits_{x\to a}\frac{g^{(m+1)}(x)}{(x-a)^{\beta}}=C\neq 0$, 则一方面, 由引理 2,

$$
\begin{aligned}
\lim_{x\to a}F(x)&=\lim_{x\to a}\frac{f(x)-T_n[f(x)]}{g(x)-T_m[g(x)]}(x-a)^{m+\beta-n-\alpha}\\
&=\lim_{x\to a}\frac{f(x)-T_n[f(x)]}{(x-a)^{n+\alpha+1}}\frac{(x-a)^{m+\beta+1}}{g(x)-T_m[g(x)]}\\
&=\frac{A}{C}\frac{\Gamma(\alpha+1)}{\Gamma(n+2+\alpha)}\frac{\Gamma(m+2+\beta)}{\Gamma(\beta+1)}
\end{aligned}
$$

另一方面, 由引理 1 又有,

$$
F(x)=\frac{f(x)-T_n[f(x)]}{g(x)-T_m[g(x)]}(x-a)^{m+\beta-n-\alpha}=\frac{m!f^{(n+1)}(\xi)}{n!g^{(m+1)}(\xi)}(x-\xi)^{n-m}(x-a)^{m+\beta-n-\alpha}
$$

其中 ξ 在 a 与 x 之间, 且当 $x\to a$ 时, 有 $\xi\to a$, 于是就又有

$$
\lim_{x\to a}F(x)=\lim_{x\to a}\frac{m!f^{(n+1)}(\xi)}{n!g^{(m+1)}(\xi)}(x-\xi)^{n-m}(x-a)^{m+\beta-n-\alpha}
$$

$$=\lim_{\xi\to a}\frac{m!f^{(m+1)}(\xi)}{n!(\xi-a)^{\alpha}}\frac{(\xi-a)^{\beta}}{g^{(m+1)}(\xi)}\lim_{x\to a}\left(\frac{x-\xi}{x-a}\right)^{n-m}\left(\frac{\xi-a}{x-a}\right)^{\alpha-\beta}$$

$$=\lim_{x\to a}\frac{m!A}{n!C}\left(\frac{x-\xi}{x-a}\right)^{n-m}\left(\frac{\xi-a}{x-a}\right)^{\alpha-\beta}$$

综上述, 就有

$$\lim_{x\to a}\left(\frac{x-\xi}{x-a}\right)^{n-m}\left(\frac{\xi-a}{x-a}\right)^{\alpha-\beta}=\frac{n!}{m!}\frac{\Gamma(\alpha+1)}{\Gamma(n+2+\alpha)}\frac{\Gamma(m+2+\beta)}{\Gamma(\beta+1)}=\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+2+\alpha)}\frac{\Gamma(m+2+\beta)}{\Gamma(m+1)\Gamma(\beta+1)}$$

由关系式 $\mathrm{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, 上式可表示为

$$\lim_{x\to a}\left(\frac{x-\xi}{x-a}\right)^{n-m}\left(\frac{\xi-a}{x-a}\right)^{\alpha-\beta}=\frac{\mathrm{B}(n+1,\alpha+1)}{\mathrm{B}(m+1,\beta+1)}$$

当 $m=n$ 时, 就有如下结论:

推论 1 设函数 $f(x)$ 和 $g(x)$ 在闭区间 $[a,b]$ 上存在 $n+1$ 阶导数, 且 $\forall x\in[a,b]$, 有 $g^{(n+1)}(x)\neq 0$, 若存在 $\alpha\geq 0$ 和 $\beta\geq 0(\alpha\neq\beta)$, 对 $\forall x\in(a,b]$ 有 $\lim\limits_{x\to a}\frac{f^{(n+1)}(x)}{(x-a)^{\alpha}}=A$ 和 $\lim\limits_{x\to a}\frac{g^{(n+1)}(x)}{(x-a)^{\beta}}=C\neq 0$. 则由 (1) 式所确定的 "中间点"$\xi$ 满足

$$\lim_{x\to a}\frac{\xi-a}{x-a}=\left(\frac{\Gamma(n+2+\beta)}{\Gamma(n+2+\alpha)}\frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)}\right)^{\frac{1}{\alpha-\beta}} \tag{5}$$

证明 由于 $m=n$ 和 $\alpha\neq\beta$, 则 $m-n=0$, 于是 (4) 可写为

$$\lim_{x\to a}\left(\frac{\xi-a}{x-a}\right)^{\alpha-\beta}=\frac{B(n+1,\alpha+1)}{B(n+1,\beta+1)}=\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+2+\alpha)}\frac{\Gamma(n+2+\beta)}{\Gamma(n+1)\Gamma(\beta+1)}=\frac{\Gamma(\alpha+1)}{\Gamma(n+2+\alpha)}\frac{\Gamma(n+2+\beta)}{\Gamma(\beta+1)}$$

所以, $\lim\limits_{x\to a}\frac{\xi-a}{x-a}=\left(\frac{\Gamma(n+2+\beta)}{\Gamma(n+2+\alpha)}\frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)}\right)^{\frac{1}{\alpha-\beta}}$

当 $\alpha=\beta$ 时, 又有如下结论:

推论 2 设函数 $f(x)$ 在闭区间 $[a,b]$ 上存在 $n+1$ 阶导数; 函数 $g(x)$ 在闭区间 $[a,b]$ 上存在 $m+1$ 阶导数 $(n\neq m)$, 且 $\forall x\in[a,b]$, $g^{(m+1)}(x)\neq 0$; 若存在 $\alpha\geq 0$, $\forall x\in(a,b]$ 有 $\lim\limits_{x\to a}\frac{f^{(n+1)}(x)}{(x-a)^{\alpha}}=A$ 和 $\lim\limits_{x\to a}\frac{g^{(m+1)}(x)}{(x-a)^{\alpha}}=C\neq 0$.

则由 (1) 式所确定的 "中间点"ξ 满足

$$\lim_{x\to a}\frac{\xi-a}{x-a}=1-\left(\frac{\Gamma(m+2+\alpha)}{\Gamma(n+2+\alpha)}\frac{\Gamma(n+1)}{\Gamma(m+1)}\right)^{\frac{1}{n-m}} \tag{6}$$

证明 由于 $\alpha=\beta$ 和 $n\neq m$, 则 $\alpha-\beta=0$, 于是 (4) 可写为

$$\lim_{x\to a}\left(\frac{x-\xi}{x-a}\right)^{n-m}=\frac{B(n+1,\alpha+1)}{B(m+1,\alpha+1)}=\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+2+\alpha)}\frac{\Gamma(m+2+\alpha)}{\Gamma(m+1)\Gamma(\alpha+1)}=\frac{\Gamma(n+1)}{\Gamma(n+2+\alpha)}\frac{\Gamma(m+2+\alpha)}{\Gamma(m+1)}$$

所以,

$$\lim_{x\to a}\frac{x-\xi}{x-a}=\left(\frac{\Gamma(m+2+\alpha)}{\Gamma(n+2+\alpha)}\frac{\Gamma(n+1)}{\Gamma(m+1)}\right)^{\frac{1}{n-m}}$$

而 $\frac{\xi-a}{x-a}=1-\frac{x-\xi}{x-a}$, 于是有 $\lim\limits_{x\to a}\frac{\xi-a}{x-a}=1-\left(\frac{\Gamma(m+2+\alpha)}{\Gamma(n+2+\alpha)}\frac{\Gamma(n+1)}{\Gamma(m+1)}\right)^{\frac{1}{n-m}}$

以上的定理和推论, 基本涵盖了最近几年关于微分中值定理和积分中值定理“中间点”渐进性方面的一些研究结果 (以下结论的证明均略去).

推论 3[2] 若函数 $f(x)$ 和 $g(x)$ 在闭区间 $[a,b]$ 上连续, 在开区间 (a,b) 内可导, 且 $\forall x\in(a,b)$, $g'(x)\neq 0$, 则由Cauchy中值定理 $\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}$ 确定的中间点 ξ 满足 $\lim\limits_{x\to a}\frac{\xi-a}{x-a}=\frac{1}{(1+\alpha)^{\frac{1}{\alpha}}}$

推论 4[3] 若函数 $f(x)$ 和 $g(x)$ 在 $[a,b]$ 上连续, $g(x)\neq 0$, $\forall x\in(a,b)$. 存在 $\alpha\geq 0,\beta\geq 0,(\alpha\neq\beta)$, 使 $\lim\limits_{x\to a}\frac{f(x)}{(x-a)^{\alpha}}=A\neq 0,\lim\limits_{x\to a}\frac{g(x)}{(x-a)^{\beta}}=B\neq 0$, 则由 $\frac{\int_a^b f(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x}=\frac{f(\xi)}{g(\xi)}$ 所确定的 ξ 有 $\lim\limits_{x\to a}\frac{\xi-a}{x-a}=\left(\frac{1+\beta}{1+\alpha}\right)^{\frac{1}{\alpha-\beta}}$.

推论 5[4] 设函数 f 在闭区间 $[a,b]$ 上存在直到 $n+1$ 阶导数, $f^{(n+1)}(x)$ 在点 a 右连续, 且 $f^{(i)}(a)=0,(i=1,2,\cdots n)$, $f^{(n+1)}(a)\neq 0$, 则Lagrange中值定理 $\frac{f(b)-f(a)}{b-a}=f'(\xi)$ 的中间点 ξ 有 $\lim\limits_{x\to a}\frac{\xi-a}{x-a}=\left(\frac{1}{1+n}\right)^{\frac{1}{n}}$

推论 6[5] 函数 $f(x)$ 在点 a 的某邻域内有直到 $n+p$ 阶导数, $f^{(n+p)}(x)$ 在点 a 连续, 且 $f^{(n+p)}(a)\neq 0$, $f^{(n+k)}(a)=0$, $k=1,2,\cdots,p\text{-}1$, 则带Lagrange余项的Taylor公式 $f(a+h)=f(a)+\frac{f'(a)}{1!}h+\frac{f''(a)}{2!}h^2+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}h^{n-1}+\frac{f^{(n)}(a+\theta h)}{n!}h^n$, $\theta\in(0,1)$ 中, 必成立 $\lim\limits_{h\to 0+}\theta=\left(\frac{n!p!}{(n+p)!}\right)^{\frac{1}{p}}$

推论 7[6-7] 如果 $f(x)$ 在 $[a,b]$ 上连续, 而且 $\lim\limits_{x\to a+}\frac{f(b)-f(a)}{(b-a)^{\alpha}}=A\neq 0$, $g(x)$ 在 $[a,b]$ 上连续且不变号, $\lim\limits_{x\to a+}\frac{g(a)}{(x-a)^{\alpha}}=B\neq 0$, ξ 是公式 $\int_a^b f(x)\mathrm{d}x=f(\xi)(b-a)$ 的中间点, 则 $\forall p\geq 0,q>0$ 有

$$\lim_{x\to a}\frac{\xi-a}{x-a}=\left(\frac{p+1}{p+q+1}\right)^{\frac{1}{p}}$$

推论 8[8-9] 设函数 $f(t)$ 在 $[a,x]$ 可导, 单调, 且 $f(t)$ 不是常数, $g(t)$ 在 $[a,x]$ 连续且不变号, η 是由积分第二中值定理 $\int_a^b f(x)g(x)\mathrm{d}x=f(a)\int_a^{\eta}g(x)\mathrm{d}x+f(b)\int_{\eta}^b g(x)\mathrm{d}x$ 所确定的“中间点”, 则

1) 当 $f'(a)\neq 0$ 时, $\lim\limits_{x\to a}\frac{\eta-a}{x-a}=\frac{1}{2}$;

2) 当 $f'(a)=\cdots=f^{(n-1)}(a)=0$, 但 $f^{(n)}(a)\neq 0$ 时, $\lim\limits_{x\to a}\frac{\eta-a}{x-a}=\frac{n}{n+1}$.

由此可见, 定理 1 是对现有渐进性结果的一个推广.

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The Asymptotic Property of "Midpoint" for Generalized Cauchy Mean Value Theorem

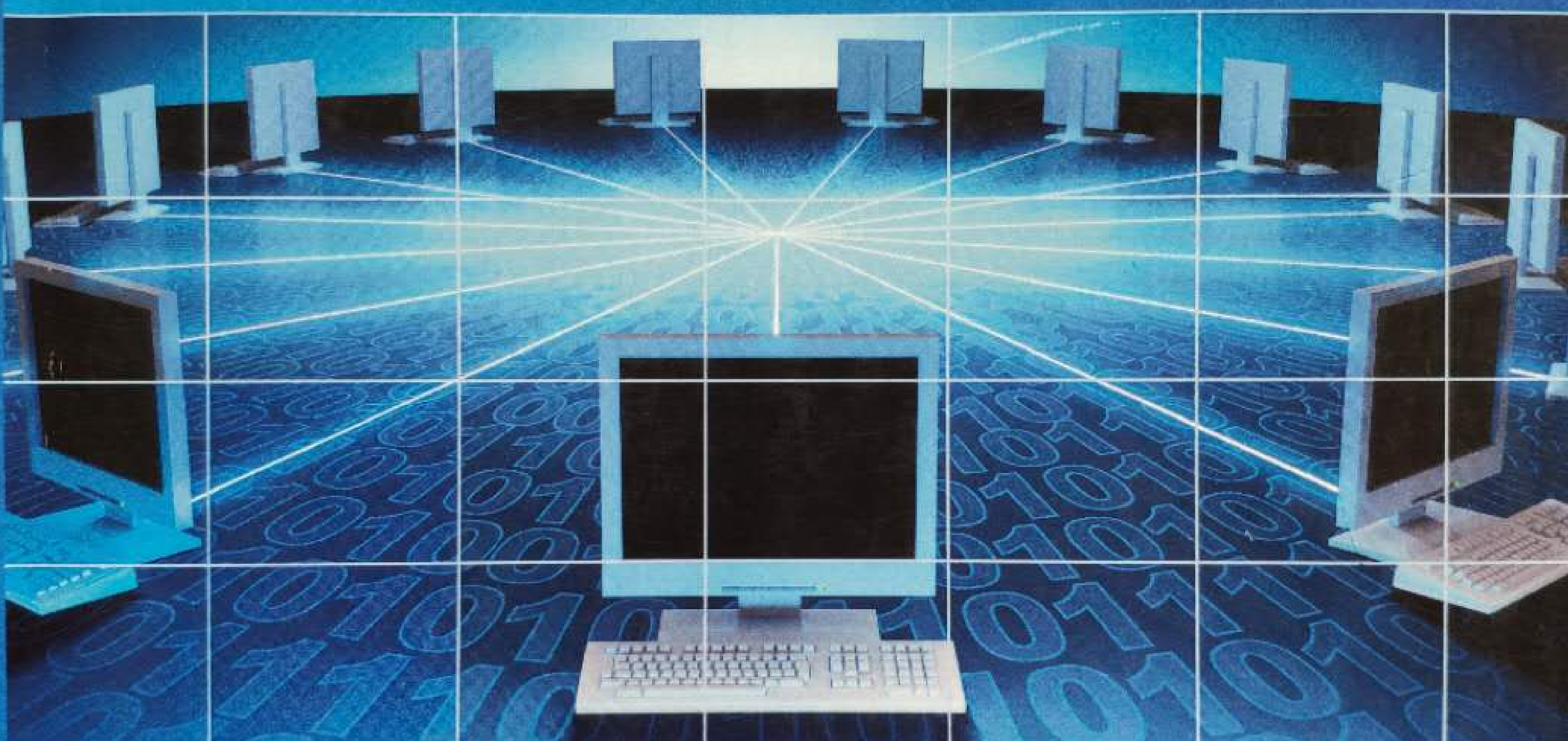
DU Zheng-guang

(Department of Mathematics, Longnan Teachers College, Chengxian 742500, China)

Abstract: By discussing the generalized Cauchy mean value theorem, obtained the asymptotic property of "midpoint" for the generalized Cauchy mean value theorem, and popularized some conclusions.

Keywords: cauchy mean value theorem; midpoint; asymptotic property

Advances in Computer Science Research



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Research on Collaborative Filtering Recommendation Algorithm Based on Matrix Decomposition Method

LiJuan

Longnan Teachers' College, Chengxian 742500,China

Keywords: Matrix decomposition, Collaborative filtering, Data mining, Least squares, Personalization, Regularization.

Abstract. In order to realize the personalized recommendation of internet mass data, according to the characteristics of internet mining data set and combined with mathematical algorithms, this paper proposes a new forecasting and computing model of adding the regularization constraint and least square method based on the traditional matrix decomposition model (SVD), improving the speed and accuracy of the proposed algorithm. Matrix decomposition before and after improvement carries out experiments and results analysis with filtering recommendation algorithm, the experimental results show that the speed and accuracy of two prediction score calculation methods have some promotion after adding the regularization constraint and the least squares. After joining the regular constraints, the RMSE values obtained by MATLAB will monotonic decrease, avoiding the over fitting phenomenon and improving the calculation quality.

Introduction

In today's Internet Era, the amount of information is growing at a constant geometry. Internet users have begun to not worry about the lack of information, but rather to worry about how to get the effective information [1,2]. Search engine is an important tool to obtain Internet data, and it also changes the situation of information explosion to a certain extent, improving the user to obtain the ability of target data in mass data [3]. However, the search engine is in the construction of key words, there are still many problems in the information extension and novelty. Personalized recommendation system can use data mining calculation, exploring the data relevance according to the relevant mathematical algorithms and providing interested content for the users, which has important significance for the mass of Internet data processing.

Information Explosion and Data Mining Recommendation Algorithm

Internet has brought a lot of information for people, but also caused the problem of information explosion, the amount of information is too large, and people do not know what information useful, which is useless, so we are unable to know what information we want [4-6]. Twitter information released ninety million in a day, the average number of users on the Facebook reached 130, while youtube uploaded video per minute also reached an average of 34. Internet has experienced a number of major experiences.

Portal era. In the portal era, the main activity of the internet is the portal site traffic, and people can query their own needs in the various categories of the portal. The portal information is provided primarily by the portal's professionals, it is similar to the traditional media and newspapers, users receives information passively because they cannot create information and provide feedback.

Search engine era. With the increasing amount of information on the internet, the demand of people has also changed, the user begins to receive information from passive to active search information, so the search engine came into being, it is an effective way to deal with the information explosion, and the most representative is Google.

Web2.0 era. In the Web2.0 era, everyone has become the internet information provider. Internet changes to read and write by read only, the supply of information has soared, including blog, SNS, Wiki and other typical internet applications. Web2.0 exacerbated the impact of information explosion, and the question of search answer reached a million, so that we cannot respond for the diversity of search results.

In the vast amount of search data, we need to use personalized recommendation system. According to the corresponding algorithm, we carry out data mining and machine learning, the use of internet users' ratings and feedback information fully tap the data features, and users can really get the data in the mass data.

Collaborative Filtering Recommendation Algorithm based on Improved Matrix Factorization

In order to improve the matrix decomposition and filtering algorithm, the traditional algorithm is analyzed in the recommendation system [7]. Suppose that the matrix of m users and n rating objects is R, the recommended object feature matrix and user feature matrix are respectively V and U. After collaborative filtering algorithm based on SVD is simplified, the input is the user's score matrix R and the characteristic number d, output is the approximation matrix X of the matrix R, in which the specific steps of the algorithm can be summarized as follows [8-10]:

(1) Data initialization, the user can obtain R_{norm} after rating matrix R carries out standardization;
(2) Determining the matrix dimension k, S is simplified as k dimension matrix, so as to get S_k;
(3) After using the SVD algorithm decomposes R_{norm}, we can get U, S and V;
(4) In accordance with the steps (3), U_k and V_k are simplified;
(5) Calculating the square root S_k, it will be recorded as $S_k^{1/2}$;
(6) According to $S_k^{1/2}$, the calculation of the relevant matrix $S_k^{1/2}V$ and $US_k^{1/2}$;
(7) For the j prediction score, the use i can be written $P(i,j) = \overline{R}_i + U_k S_k^{1/2}(i) S_k^{1/2} V_k(j)$.

By using the above algorithm, the user can get the prediction score for any project, in which $\overline{R}_i$ is the average value in all evaluation item users. In order to improve the algorithm, we can find a low rank matrix X to carry on the maximum extent approximation on R. The minimize Frobenius loss function is

$$L(x) = \sum_{ij} (R_{ij} - X_{i,j})^2 \tag{1}$$

Among them, $L(x)$ shows the objective function, $(R_{ij}-X_{ij})^2$ shows the square error term of low rank approximation, and the improved algorithm can be solved quickly on $L(x)$. In the recommendation system, in order to realize the algorithm, the formula (1) is rewritten

$$L(U,V) = \sum_{ij} (R_{ij} - U_i V_j^T)^2 \tag{2}$$

In order to prevent the fitting problem in the calculation process, the formula (2) is a regularization constraint, and the formula (2) can be rewritten as

$$L(U,V) = \sum_{ij} (R_{ij} - U_i V_j^T)^2 + \lambda(\|U_i\|^2 F + \|V_j\|^2 F) \tag{3}$$

V is fixed, we can solve U_i formula on U_i derivation, it is

$$U_i = R_i V_{ui} (V_{ui}^T V_{ui} + \lambda n_{ui} I)^{-1}, i \in [1 \quad m] \tag{4}$$

Among them, R_i shows the film composition vector after users i are rated, V_{ui} shows the user evaluation of film feature vector matrix, n_{ui} indicates the number of film comment. Similarly, V_i will be fixed and derivate, it can get

$$V_j = R_j U_{mj} (U_{mj}^T U_{mj} + \lambda n_{mj} I)^{-1}, j \in [1 \quad m] \tag{5}$$

Among them, R_j represents the film composition vector after user j rating, U_{mj} represents the user i film review feature vector matrix, n_{mj} represents the number of film reviews. In formula (4) and formula (5), I shows d*d unit matrix, the regularization cross can use the least square algorithm to continue to optimize. First of all, we use the Gauss random number initialization matrix of mean value 0 and deviation 0.01, and then the use of formula (4) updates U, the use of formula (5) updates V, until the calculation value of RMSE is convergence, or the number of iterations is enough, the calculation can stop.

The Experimental Results and Analysis of Improved Recommendation Algorithm

This algorithm uses MATLAB software to process the MovieLens data set, and the data set is created and maintained by the GroupLens research group of America Minnesota University [11]. The data packet contains 100000 scoring record of 900 users for the 1600 films.

Experiments are divided into three groups that are traditional SVD algorithm, joining regularization constraint algorithm, joining regularization constraint algorithm and least squares algorithm experiment [12,13]. Each algorithm uses the configuration same computer, the computer is in the same local area network. In the experiment, the number of iterations is 80, and the number of features is 1-9. The results obtained by calculation are shown in Figure 1.

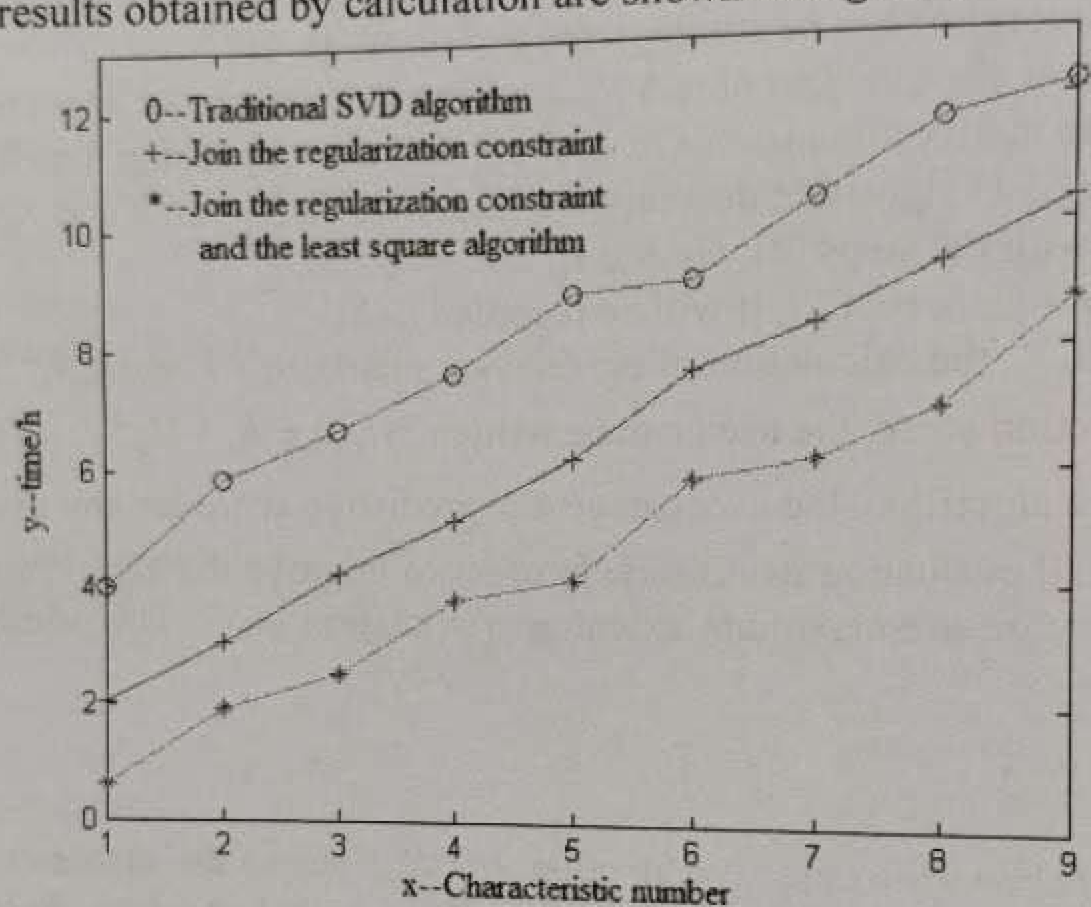

Fig. 1 The running time comparison of different algorithms

Figure 1 shows the comparison results of the three experimental group running time, in which the horizontal axis is the number of features, the vertical coordinates is the running time. The graph can be seen that after adding a regular constraint and the least squares algorithm, running time is significantly shortened, and with the increase of the number of features, this effect is more obvious.

Table 1 shows the accuracy of different algorithms, it can be seen that the improved algorithm accuracy is obviously higher than the traditional algorithm accuracy, and simultaneously adding the regular and least square algorithm are more accurate than join the algorithm accuracy of regular constraint alone.

Table 1. Comparison of the traditional SVD algorithm and the improved algorithm calculation accuracy

Experiment number	Traditional SVD algorithm calculation accuracy	Join regularization constraint calculation accuracy	Join the regular and the least squares method calculation accuracy
1	0.913	0.952	0.992
2	0.912	0.956	0.995
3	0.925	0.962	0.987
4	0.918	0.953	0.996
5	0.9163	0.966	0.988

In order to further study the effect of the algorithm, the results of the experiments use RMSE evaluation criteria, RMSE goes through the calculation forecast deviation between user score and actual score to measure the accuracy of the forecast, which is most commonly measurement method of recommended level. The smaller the value of RMSE, the higher the quality of the recommendation, hypothesis that the prediction score vector of N project is $\{p_1, p_2, \cdots, p_N\}$, and the actual user's score vector is $\{r_1, r_2, \cdots, r_N\}$, the RMSE calculation of the algorithm is

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(p_i - r_i)^2}{N}} \tag{6}$$

Using MATLAB software calculates the traditional algorithm and the RMSE value of improved SVD algorithm, and we can get the calculation of the final results as shown in Figure 2.

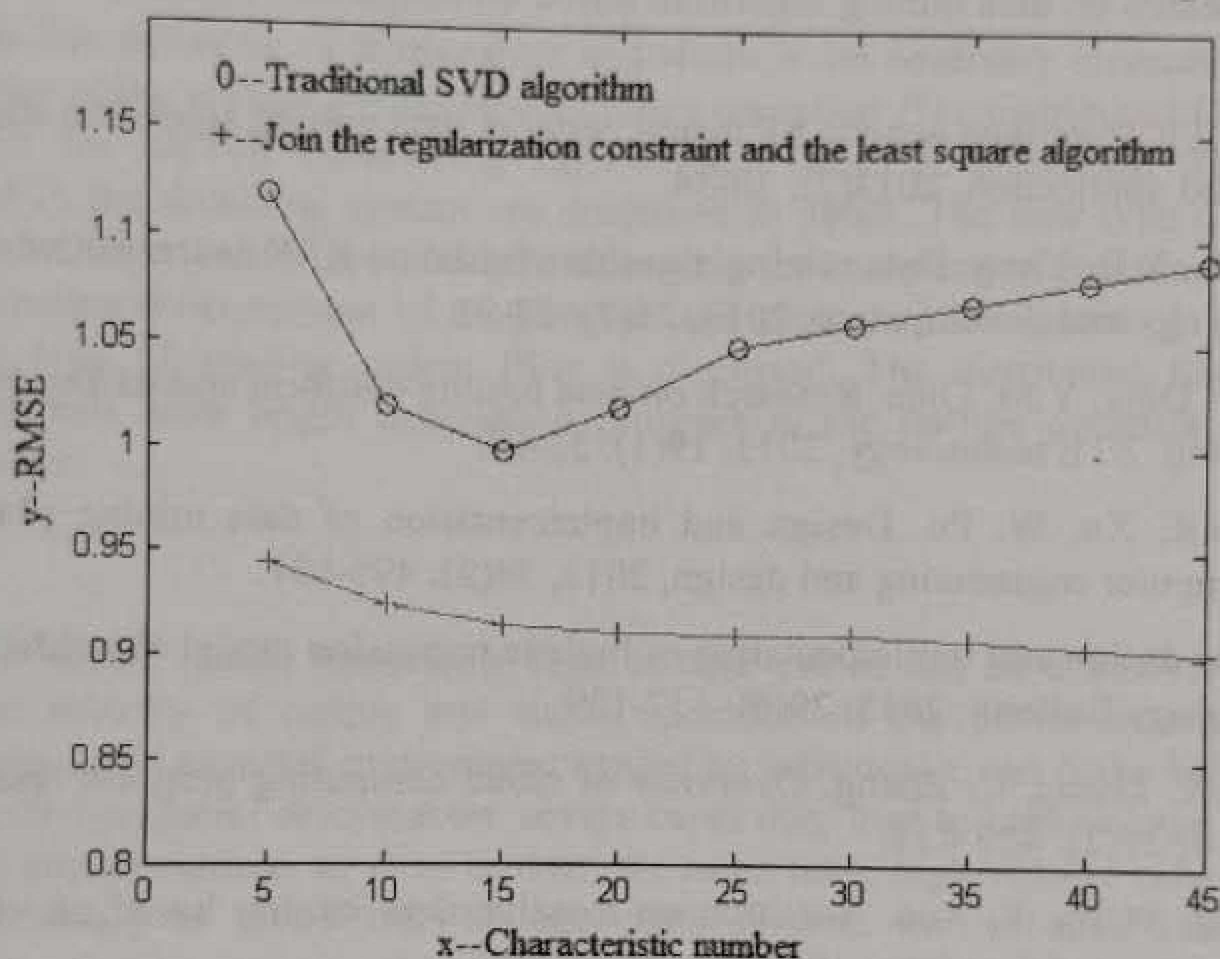

Fig. 2 The performance comparison of improved algorithm

Figure 2 shows the algorithm performance comparison before and after the improvement, the algorithm can be seen that the improved algorithm amplitude is better than the original algorithm amplitude under each characteristic. With the increase of the number of features, the RMSE value of the improved algorithm monotonic decreases, the feature matrix U and V are constrained after the reason of the decrease is mainly due to the addition of the regularization constraint, so as to avoid the occurrence of over fitting phenomenon. However, the traditional algorithm is not added constraint, and RMSE will be increased with the increase of the number of features, so the fitting phenomenon has been affected the quality of the calculation.

Summary

According to the characteristics of internet search engine data mining, this paper joins the regularization constraint and the least square algorithm based on the traditional matrix decomposition model (SVD) and designs a new internet data personalized recommendation system. In order to verify the reliability of the system, before and after the improved algorithm is carried out experiments and results analysis, the experiment uses MATLAB software on MovieLens data processing. Under the calculation of different characteristic numbers, the time and precision of different algorithms are calculated by MATLAB iterative calculation. The experimental results show that before and after the improved algorithm can effectively improve the speed and accuracy of prediction score calculation, which can get a better RMSE value.

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Saddle Point Non-Singular Value Solution Based on Generalized Inverse Hermitian Triangulation and Split Iteration

LiJuan

Longnan Teachers College,Longnan 742500,China

Keywords: Saddle point system, Split iteration, Generalized anti Hermitian, Convergence, Non-singular.

Abstract. In order to achieve the nonsingular solution of saddle point linear system, this paper proposes an improved nonlinear Uzawa splitting iteration method, and combined with the generalized anti Hermitian triangle method, the algorithm is further modified, to improves the convergence of the algorithm. Finally, using the GMRES method and block diagonal Hermitian constraints carry out solve for the saddle point linear system, and then this paper can obtain the comparison results of saddle point coefficient matrix spectrum and calculation speed and accuracy. It can be seen from the calculation results that combined with the generalized inverse Hermitian triangulation method, the modified nonlinear Uzawa splitting iteration method has faster calculation speed and better convergence.

Introduction

In many complex scientific and engineering fields, the use of large sparse linear systems will generate a saddle point linear structure, which has great significance for the study of saddle point linear systems singular solutions [1]. For solving the saddle point linear system, many iterative algorithms and preconditioning algorithm are used in the saddle point solution method, they also made a certain effect, but these methods are often aimed at saddle point nonsingular solution [2]. How to deal with the non singular saddle point problem, there are better algorithm efficiency and accuracy, and combined with the stoke equations of computational fluid dynamics, this paper will do further research.

Saddle Point Structure Computational Fluid Linear System

In many engineering and science areas, there exists the saddle point structure of linear systems, in which mainly includes the constraint conditions optimization, control optimization, weighted least square optimization, numerical and fluid calculation optimization [3]. Saddle point linear system is generated by discrete Stokes equation or the second order elliptic equation, and generating process is generally produced in the use of the finite difference or finite element process [4]. In order to calculate Stoke equations as an example in the circulation, this study introduces the saddle point system. The hypothesis Ω is R^2 or R^3 to solve a range and the range is connected region. The use of the constraint condition on $\partial\Omega$ boundary, given field function is f, we can solve force field function p and velocity field function u by the formula (1).

$$\begin{aligned} &\alpha u_t - v\Delta u + (u.\nabla)u + \nabla p = f, \quad in\Omega \\ &-\nabla \cdot u = 0, \quad in\Omega \end{aligned} \tag{1}$$

Wherein v represents the viscosity coefficient. When $\alpha = 0$, the formula (1) can be written in the steady-state Stokes equation.

$$\begin{aligned} &-v\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad in\Omega \\ &-\nabla \cdot u = 0, \quad in\Omega \end{aligned} \tag{2}$$

If only considering the formula (2), assuming an arbitrary initial vector $u^{(0)}$ in the case of $m=0,1,2,...$, carries out the iterative calculation for the formula (2). The iterative method selects the Picard form, we can get

$$\begin{aligned} -v\Delta u^{(m+1)}+(u^{(m)}\cdot\nabla)u^{(m+1)}+\nabla p^{(m+1)}&=f,\quad in\Omega \\ -\nabla\cdot u^{(m+1)}&=0,\quad in\Omega \end{aligned} \tag{3}$$

In formula (3), the vector w is known field, and the field is Picard iteration and divergence free velocity field. If meeting $w=0$ conditions, using finite element or finite difference carry out discrete for the formula (3), then we can get the saddle point linear system.

$$Au\equiv\begin{pmatrix} M & E \\ -E^{*} & 0\end{pmatrix}\begin{pmatrix} u_1 \\ u_2\end{pmatrix}=\begin{pmatrix} f_1 \\ f_2\end{pmatrix}\equiv f. \tag{4}$$

Wherein $M\in C^{p\times p}$ represents the Hermitian positive definite matrix; $E\in C^{p\times p}$ represents that matrix satisfies $q\le p$; $f\in C^{p+q}$ represents matrix coefficient and shows the known vector in the computational fluid space numerical region, it satisfies the condition $A\in C^{(p+q)\times(p+q)}$, and there are $A\in C^{p}$ and $f_2\in C^{q}$.

The Saddle Point Singular Value Solution Based on the Generalized Inverse Hermitian Triangulation and Split Iteration

This research mainly adopts split iteration method and generalized anti Hermitian triangle method, to calculate the singular solutions of saddle point [5]. In order to study the generalized saddle point problem, we establish the mathematical model of saddle point problems. The formula is as follows:

$$Au=\begin{pmatrix} A & B^T \\ B & -C\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}=\begin{pmatrix} f \\ g\end{pmatrix}=b. \tag{5}$$

Wherein $A\in R^{n\times n}$ has positive definite, and $C\in R^{m\times m}$ has positive semi definite. For $B\in R^{m\times n}, m\le n$, in which B^T is the transpose matrix of B. For $f\in R^{n}, g\in R^{m}$, the system is very important for a class of numerical calculation linear system, we can use the Uzawa algorithm of iterative splitting to solve the problem. The assumption of any given $x_0\in R^{n}$ and $y_0\in R^{m}$, the sequence of iterations $\{(x_i,y_i)\}(i=0,1,...,)$ is defined as

$$\begin{cases} x_{i+1}=x_i+A^{-1}(f-Ax_i-B^Ty_i) \\ y_{i+1}=y_i+TQ_B^{-1}(Bx_{i+1}-Cy_i-g)\end{cases}. \tag{6}$$

Wherein τ represents real parameters, Q_B represents positive definite matrix, and its satisfaction is

$$\gamma(Q_Bw,w)\le((B(A)^{-1}B^T+C)w,w)\le(Q_Bw,w),\forall w\in R^{m}. \tag{7}$$

The error of the iterative calculation is

$$e_i^x=x-x_i, e_i^y=y-y_i. \tag{8}$$

For the non symmetric saddle point problems, combined with the generalized anti Hermitian triangle method, we consider the different constraint conditions, such as block diagonal or triangular block, we can expand the calculation conditions and constraint conditions. Finally, through the pre-processing matrix spectrum, this paper is to characterize the saddle point of the singular solution.

To consider the following diagonal block constraints, there is

$$\varsigma=\begin{pmatrix} G & 0 \\ 0 & CG^{-1}B^T\end{pmatrix}. \tag{9}$$

Wherein $G-E=A$ represents a non singular split matrix, which is a block diagonal pretreatment matrix eigenvalue analysis. Its constraints can also be written

$$P=\begin{pmatrix} G & 0 \\ 0 & -CG^{-1}B^{T} \end{pmatrix}. \tag{10}$$

In the case of $G=A$, we can get the exact block constraints. This paper presents a very similar constraint with the previous introduction.

$$P_{ab}=\begin{pmatrix} G & aB^{T} \\ 0 & bCG^{-1}B^{T} \end{pmatrix}. \tag{11}$$

The constraint is based on generalized anti Hermitian triangle method, it not only can precise processing matrix characteristic value, but also can handle the perturbations of non exact constraint conditions, and its calculation results is better than above method.

Saddle Point Generalized Anti Hermitian Triangle Method and Iterative Splitting Solution

For solving the saddle point system, it can be combined with generalized inverse Hermitian triangulation method and iterative split Uzawa algorithm. Considering different constraints, it can go through the pretreatment matrix spectrum to characterize the singular solutions of saddle point [6-8]. Combined with the constraint conditions GMRES method, the examples can solve the saddle point system, in which the saddle point system is

$$Ax=b. \tag{12}$$

Among them, assuming that the values of b can make the values of x is 1, we can use zero vectors as the initial vector, the iteration termination condition is set to calculate the number of more than 50, and the convergence precision is set to

$$\frac{\|b-Axk\|_2}{\|b\|_2}\le 10^{-6}. \tag{13}$$

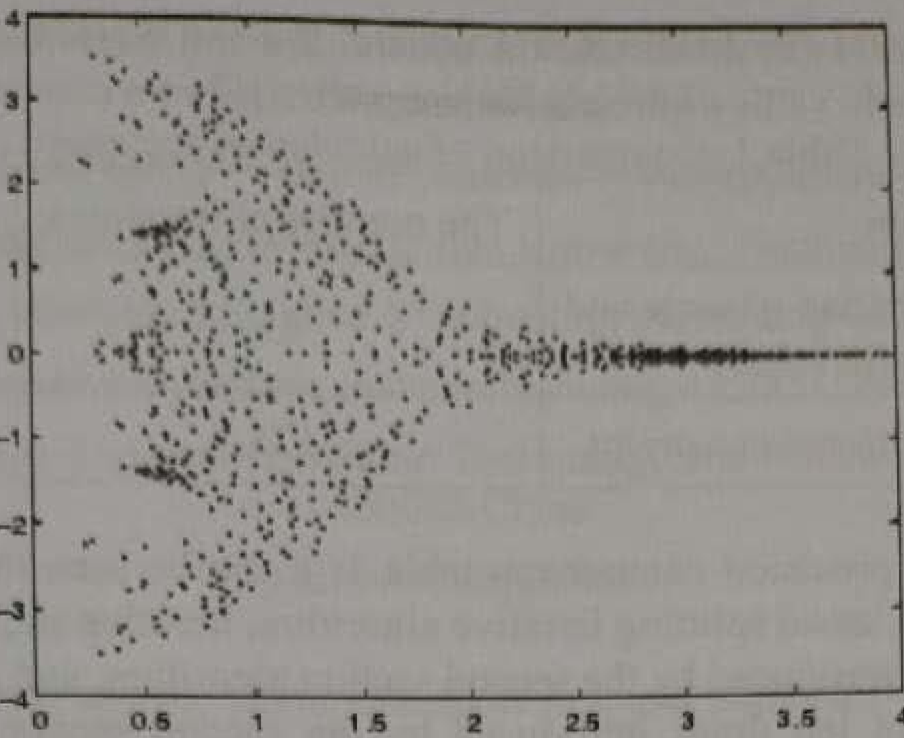
Considering the example has appropriate boundary conditions, we can convert it to Oseen problem.

$$\begin{cases} -v\Delta u+w\cdot\nabla u+\nabla p=f, in\Omega \\ divu=0, in\Omega \end{cases} \tag{14}$$

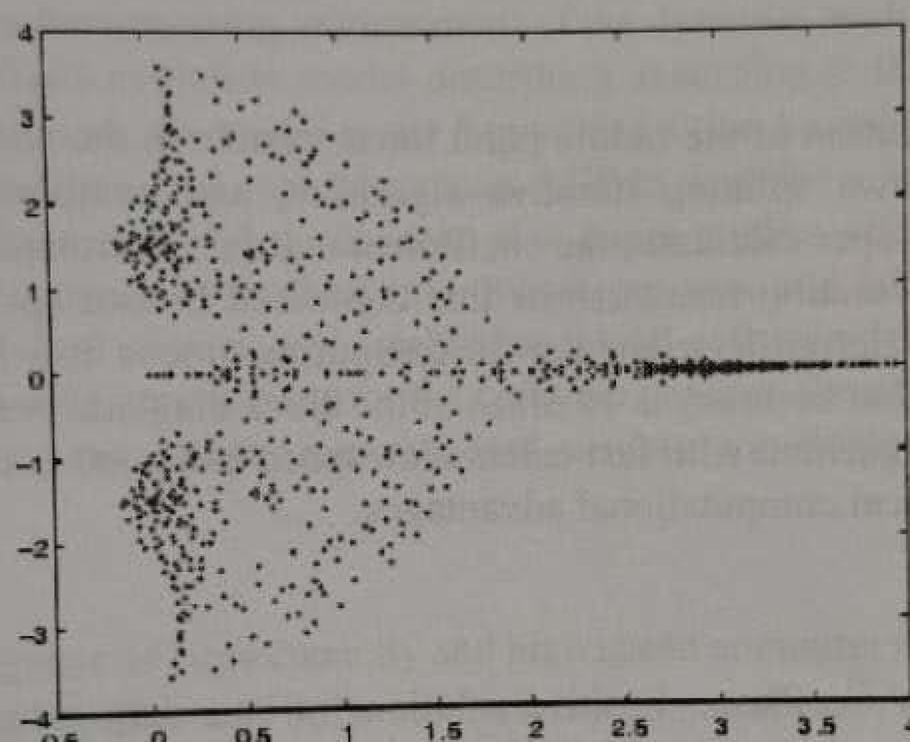
Wherein the vector function u is expressed in the Ω upper speed, and w meets $\nabla\cdot w=0$. Using square element $n\times n$ grid and discrete solution method, we can obtain matrix $\hat{A}$.

$$\hat{A}=\begin{pmatrix} F1 & & B_u^T \\ & F2 & B_v^T \\ Bu & Bv & 0 \end{pmatrix}. \tag{15}$$

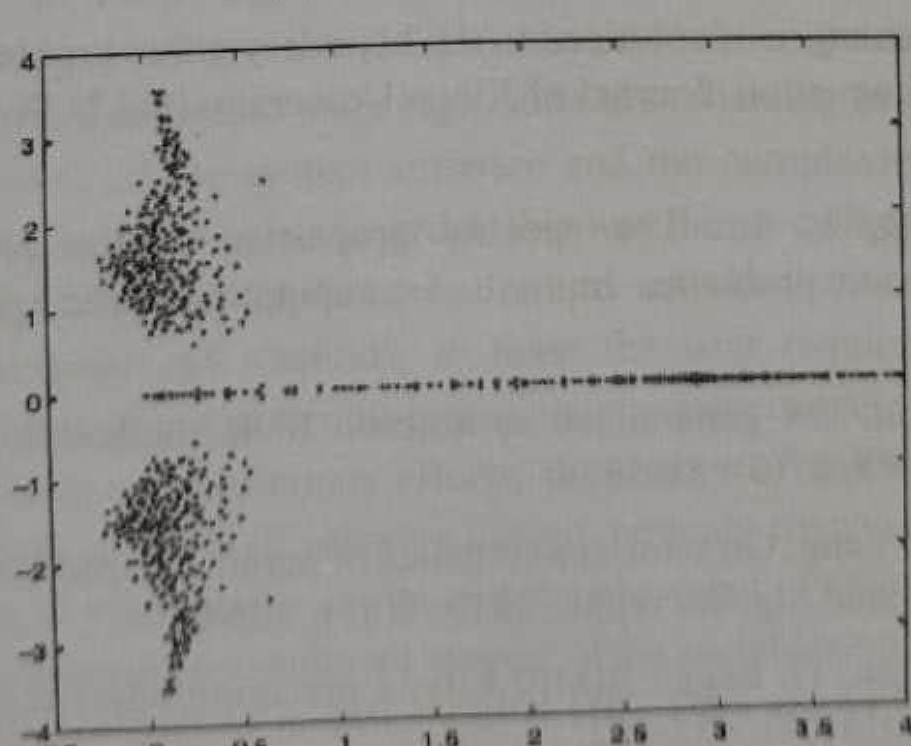
Among them, $\begin{pmatrix} F_1 & \\ & F_2 \end{pmatrix}=A$ has positive definite. This calculation uses the 18×18 grid, we can structure matrix $\hat{A}$ through the A. The saddle point coefficient matrix is obtained by the calculation as shown in Figure 1.



(a) Nullity(A)=0



(b) Nullity (B)=0.5m



(c) Nullity (C)=5m

Fig. 1 Saddle point coefficient matrix spectra

Figure 1 shows the matrix spectra of the saddle point coefficients obtained by calculating. In Figure 1, we can calculate the spectral distribution of the original matrix spectrum *Ai* for a matrix *A* with

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具有非局部源的退化奇异抛物方程组解的爆破

王胜青 [1], 何万生 [2], 彭聪明 [2,3]

(1. 陇南师范高等专科学校数信学院, 甘肃 陇南 742500;

2. 天水师范学院数学与统计学院, 甘肃 天水 741000; 3. 兰州大学数学与统计学院, 甘肃 兰州 730000)

摘要：研究了一类新的包含幂函数和指数函数相耦合的具有非局部源的抛物方程组. 用正则化的方法证明了局部解的存在唯一性, 用上下解方法得到了整体存在和在有限时刻爆破的充分条件.

关键词：退化; 整体存在性; 爆破

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1 引言

本文考虑了下述带非局部源的退化奇异反应扩散方程组:

$$\begin{cases} u_t-(x^{\alpha}u_x)_x=\int_0^a v^n(x,t)\mathrm{e}^{mu(x,t)}\mathrm{d}x,\ (x,t)\in(0,a)\times(0,T), \\ v_t-(x^{\beta}v_x)_x=\int_0^a u^p(x,t)\mathrm{e}^{qv(x,t)}\mathrm{d}x,\ (x,t)\in(0,a)\times(0,T), \\ u(0,t)=u(a,t)=v(0,t)=v(a,t)=0,\ t\in(0,T), \\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x),\ x\in[0,a]. \end{cases} \tag{1.1}$$

其中 $u_0(x),v_0(x)\in C^{2+\gamma}(\overline{D})(\ \gamma\in(0,1))$ 是非负非平凡函数. m,n,p,q 是正实数,

$$u_0(0)=u_0(a)=v_0(0)=v_0(a)=0,$$

u_0 及 v_0 满足相容性条件, $T>0,a>0,\alpha,\beta\in[0,2)$.

设 $D=(0,a),\Omega_t=D\times(0,t]$, $\overline{D}$ 及 $\overline{\Omega}_t$ 分别是它们的闭包. 当 x 趋于 0 时 u_x,u_{xx} 和 v_x,v_{xx} 的系数可能趋于 0 或 ∞, 故方程是退化奇异的.

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作者简介：王胜青 (1965-), 硕士, 副教授, 研究方向：概率论与数理统计、微分方程.

该问题是一个描述热传导过程的数学模型, 具体可参考文献 [1] . 当 $\alpha=\beta=0$ 时, (1.1) 是一个具有非局部源的半线性热方程. 具有局部源 $f(u)$ 的热方程已经受到很多关注. 最主要的工作是由 Friedman 和文献 [2-4] 的作者所做. 对非局部源的首个 Fujita- 型结果是由 Galaktionov 和 Levine 在文献 [5] 中得到的. 文献 [2-3] 得到解爆破的一般条件, 证明了爆破集是区域的一个紧子集, 而且得到了爆破解的渐进行为. 最近，带非局部源热方程的研究引起了很多关注. 关于 (1.1), 当 $\alpha=\beta=0$ 时, 文献 [6-9] 都进行了研究. 在文献 [10] 中, 作者研究了下述问题:

$$\begin{cases} u_t-(x^\alpha u_x)_x=\int_0^a f(u(x,t))\mathrm{d}x,\ (x,t)\in(0,a)\times(0,T), \\ u(0,t)=u(a,t)=0,\ t\in(0,T), \\ u(x,0)=u_0(x),\ x\in[0,a], \end{cases} \tag{1.2}$$

得到了解的局部存在唯一性. 在适当条件下, 还研究了正解的整体存在性和解在有限时刻的爆破. 他们还得到了解的爆破集是整个区域. 本文推广了文献 [7,10] 的结果. 但由于退化性的出现，我们用了和文献 [7] 不同的方法. 在第二部分中, 建立了比较原理及解的存在唯一性. 关于解在有限时刻的爆破将在第三部分中讨论.

2 局部存在性

首先，我们给出 (1.1) 的上解的定义. $\gamma\in(0,1)$, $u_0(x),v_0(x)\in C^{2+\gamma}(\overline{D})$ 为非负非平凡函数. m,n,p,q 是正实数. $u_0(0)=u_0(a)=v_0(0)=v_0(a)=0$,$u_0$ 和 v_0 满足相容性条件,$T>0,a>0,\alpha,\beta\in[0,2)$.

定义 2.1 一对非负函数 $(\bar{u}(x,t),\bar{v}(x,t))$ 称为方程组 (1.1) 的上解, 如果

$$(\overline{u}(x,t),\overline{v}(x,t))\in(C([0,a]\times[0,T))\cap C^{2,1}((0,a)\times(0,T)))^2$$

而且满足

$$\begin{cases} \bar{u}_t-(x^\alpha\bar{u}_x)_x\geq\int_0^a\bar{v}^n(x,t)\mathrm{e}^{m\bar{u}(x,t)}\mathrm{d}x,\ (x,t)\in(0,a)\times(0,T), \\ \bar{v}_t-(x^\beta\bar{v}_x)_x\geq\int_0^a\bar{u}^p(x,t)\mathrm{e}^{q\bar{v}(x,t)}\mathrm{d}x,\ (x,t)\in(0,a)\times(0,T), \\ \bar{u}(0,t)\geq0,\bar{u}(a,t)\geq0,\ t\in(0,T) \\ \bar{v}(0,t)\geq0,\bar{v}(a,t)\geq0,\ t\in(0,T), \\ \bar{u}(x,0)\geq\bar{u}_0(x),\ \ \bar{v}(x,0)\geq\bar{v}_0(x),\ x\in[0,a]. \end{cases} \tag{2.1}$$

类似地,

$$(\underline{u}(x,t),\underline{v}(x,t))\in(C([0,a]\times[0,T))\cap C^{2,1}((0,a)\times(0,T)))^2$$

称为下解，如果 (2.1) 中的所有反向不等号成立.

为了证明问题 (1.1) 的正解的存在唯一性, 必须构建下述比较原理:

引理 2.1 设 $r\in(0,T)$, $c_i(x,t),d_i(x,t)(i=1,2)$ 为定义在 $[0,a]\times[0,r]$ 上的连续非负函数, 令 $(u(x,t),v(x,t))\in(C(\overline{\Omega}_r)\cap C^{2,1}(\Omega_r))^2$ 满足:

$$\begin{cases} u_t-(x^\alpha u_x)_x\geq\int_0^a[c_1(x,t)u(x,t)+c_2(x,t)v(x,t)]\mathrm{d}x,\ (x,t)\in(0,a)\times(0,r],\\ v_t-(x^\beta v_x)_x\geq\int_0^a[d_1(x,t)u(x,t)+d_2(x,t)v(x,t)]\mathrm{d}x,\ (x,t)\in(0,a)\times(0,r],\\ u(0,t)\geq0,\ \ u(a,t)\geq0,\ \ v(0,t)\geq0,\ \ v(a,t)\geq0,\ t\in(0,r],\\ u(x,0)\geq0,\ \ v(x,0)\geq0,\ x\in[0,a]. \end{cases}$$

则在 $[0,a]\times[0,T)$ 上, $u(x,t)\geq0,v(x,t)\geq0$.

证明 证明类似于文献 [11], 引理 2.1.

引理 2.2 设 $(u(x,t),v(x,t))$ 是问题 (1.1) 的非负解. 假设一对函数

$$(w(x,t),z(x,t))\in(C(\overline{\Omega}_r)\cap C^{2,1}(\Omega_r))^2$$

满足

$$\begin{cases} w_t-(x^\alpha w_x)_x\geq(\leq)\int_0^a z^n(x,t)\mathrm{e}^{mw(x,t)}\mathrm{d}x,\ (x,t)\in(0,a)\times(0,r],\\ z_t-(x^\beta z_x)_x\geq(\leq)\int_0^a w^p(x,t)\mathrm{e}^{pz(x,t)}\mathrm{d}x,\ (x,t)\in(0,a)\times(0,r],\\ w(0,t)\geq(=)0,\ \ w(a,t)\geq(=)0,\ t\in(0,r],\\ z(0,t)\geq(=)0,\ \ z(a,t)\geq(=)0,\ t\in(0,r],\\ w(x,0)\geq(\leq)u_0(x),\ \ z(x,0)\geq(\leq)v_0(x),\ x\in[0,a]. \end{cases}\tag{2.2}$$

则在 $[0,a]\times[0,T)$ 上, $(w(x,t),z(x,t))\geq(\leq)(u(x,t),v(x,t))$.

证明 只考虑 “$\geq$” 的情形 (因为对情形 “$\leq$” 证明类似). 令

$$\varphi_1(x,t)=w(x,t)-u(x,t),\varphi_2(x,t)=z(x,t)-v(x,t).$$

用 (2.2) 减去 (1.1) 并利用中值定理, 可得

$$\begin{cases} \varphi_{1t}-(x^\alpha\varphi_{1x})_x\geq\int_0^a[f_1(x,t)\varphi_1(x,t)+f_2(x,t)\varphi_2(x,t)]\mathrm{d}x,\ (x,t)\in(0,a)\times(0,r],\\ \varphi_{2t}-(x^\beta\varphi_{2x})_x\geq\int_0^a[g_1(x,t)\varphi_1(x,t)+g_2(x,t)\varphi_2(x,t)]\mathrm{d}x,\ (x,t)\in(0,a)\times(0,r],\\ \varphi_1(0,t)\geq0,\ \ \varphi_1(a,t)\geq0,\ t\in(0,r],\\ \varphi_2(0,t)\geq0,\ \ \varphi_2(a,t)\geq0,\ t\in(0,r],\\ \varphi_1(x,0)\geq0,\ \ \varphi_1(x,0)\geq0,\ x\in[0,a], \end{cases}$$

其中 $r\in(0,T)$, f_1,f_2,g_1,g_2, 是定义在 $[0,a]\times[0,r]$ 上的连续非负函数. 故引理 2.1 保证了 $(\varphi_1(x,t),\varphi_2(x,t))\geq(0,0)$, 也就是说, 在 $[0,a]\times[0,T)$ 上,

$$(w(x,t),z(x,t))\geq(u(x,t),v(x,t)).$$

显然, $(0,0)$ 是问题 (1.1) 的下解, 我们还需要构造一个上解.

引理 2.3 设存在正常数 $t_0(t_0<T)$ 使得问题 (1.1) 有上解 $(f(t),g(t))\in(C[0,t_0])^2$.

证明 令 $(f(t),g(t))$ 是 Cauchy 问题

$$\begin{cases} f'(t)=ag^n(t)\mathrm{e}^{mf(t)}, \quad g'(t)=af^p(t)\mathrm{e}^{pg(t)}, \quad t>0 \\ f(0)=a_0>0, \quad g(0)=b_0>0, \end{cases}$$

唯一的解, 其中

$$a_0=\max_{x\in[0,a]}u_0(x), \quad b_0=\max_{x\in[0,a]}v_0(x).$$

则存在正常数 t_0 使得 $(f(t),g(t))\in(C[0,t_0])^2$. 由引理 2.1, 我们知道 $(f(t),g(t))$ 是问题 (1.1) 的上解.

为了得到解的存在唯一性，需要一个正则化过程，但该过程是标准的，所以我们直接给出解的存在唯一性定理:

定理 2.1 存在 $t_0(<T)$ 使得问题 (1.1) 有唯一的非负解

$$(u(x,t),v(x,t))\in(C(\overline{\Omega}_{t_0})\cap C^{2,1}(\Omega_{t_0}))^2,$$

其中 $\alpha,\beta\in[0,2)$.

证明 证明类似于文献 [12], 定理 2.5.

定理 2.2 设 T 是使得问题 (1.1) 存在唯一解 $(u(x,t),v(x,t))\in(C(\overline{\Omega}_{t_0})\cap C^{2,1}(\Omega_{t_0}))^2$ 的 t_0 的上确界. 则问题 (1.1) 有唯一的非负解

$$(u(x,t),v(x,t))\in(C([0,a]\times[0,T))\cap C^{2,1}((0,a)\times(0,T)))^2.$$

若 $T<+\infty$, 则

$$\limsup_{t\to T}\max_{x\in[0,a]}(|u(x,t)|+|v(x,t)|)=+\infty.$$

证明 证明类似于文献 [13] 的定理 2.5.

3 整体存在性及爆破

下面考虑问题 (1.1) 解的整体存在性及在有限时刻的爆破. 主要结论如下:

定理 3.1 a) 若 $np>1$ 则问题 (1.1) 在任意区域 $(0,a)$ 上对小初值 (u_0,v_0) 存在整体解.

b）若 $np\le 1$ 且 a 适当小，则问题 (1.1) 对小初值 (u_0,v_0) 存在整体解.

证明 设 $\varphi_1(x)$ 和 $\psi_1(x)$ 分别是下述两椭圆问题:

$$\begin{cases} -(x^\alpha\varphi_1'(x))'=1,\ x\in(0,a), \\ \varphi_2(0)=\varphi_2(a)=0, \end{cases} \tag{3.1}$$

$$\begin{cases} -(x^{\beta}\psi_1'(x))' = 1,\ x \in (0,a), \\ \psi_2(0) = \psi_2(a) = 0, \end{cases} \tag{3.2}$$

的唯一解. 直接计算可得

$$\varphi_1(x) = \frac{x^{1-\alpha}(a-x)}{2-\alpha}, \quad \psi_1(x) = \frac{x^{1-\beta}(a-x)}{2-\beta}$$

而且存在正常数 C_1 和 C_2 使得

$$0 \le \varphi_2(x) \le C_1, \quad 0 \le \psi_2(x) \le C_2.$$

令

$$K = \max\left\{ \sup_{x\in(0,a)} (\delta + \varphi_1(x)), \sup_{x\in(0,a)} (\delta + \psi_1(x)) \right\}$$

其中 $\delta > 0$ 是一个常数.

对于 a) 注意到 $np > 1$, 易知存在两正常数 b_1, b_2 使得

$$b_1 \ge ab_2^n(\delta + \psi_1(x))^n \mathrm{e}^{mb_1(\delta+\varphi_1(x))},\ \ b_2 \ge ab_1^q(\delta + \varphi_1(x))^q \mathrm{e}^{pb_2(\delta+\psi_1(x))}. \tag{3.3}$$

事实上只要证明

$$b_1 \ge ab_2^n K^n \mathrm{e}^{mb_1K},\ \ b_2 \ge ab_1^p K^p \mathrm{e}^{qb_2K}. \tag{3.4}$$

选择 $b_2^n = \frac{b_1}{aK^n\mathrm{e}^{mb_1K}}$, 可以证明存在 $b_1 > 0$ 使得

$$\frac{1}{aK^n\mathrm{e}^{mb_1K}} \ge a^n b_1^{np-1} K^{np} \mathrm{e}^{nqb_2K} = h(a).$$

但是注意到当 $np > 1$ 时

$$\lim_{a\to 0+} h(a) = 0,$$

故 (3.4) 成立.

设

$$\overline{u}(x,t) = b_1(\delta + \varphi_1(x)), \overline{v}(x,t) = b_2(\delta + \psi_1(x)),$$

则有

$$\overline{u}_t \ge (x^{\alpha}\overline{u}_x)_x + \int_0^a \overline{v}^n \mathrm{e}^{m\overline{u}}\mathrm{d}x,\ \ \overline{v}_t \ge (x^{\beta}\overline{v}_x)_x + \int_0^a \overline{u}^q \mathrm{e}^{p\overline{v}}\mathrm{d}x.$$

由引理 2.2, 得到如果 $u_0(x) \le b_1(\delta + \varphi_1(x))$ 且 $v_0(x) \le b_2(\delta + \psi_1(x))$ 则 $(u,v) \le (\overline{u}, \overline{v})$. 因此, (u,v) 整体存在.

对于 b) 注意到对任意的 $n > 0, q > 0$, 若 K 充分小则 (3.4) 对某 $b_1 > 0, b_2 > 0$ 成立. 事实上, 若

$$\max\{K^n\mathrm{e}^{mK}, K^p\mathrm{e}^{qK}\} \le 1$$

则存在 $0 < b_1, b_2 \le 1$ 使得

$$b_1 \ge K^n \mathrm{e}^{mK}, b_2 \ge K^p \mathrm{e}^{qK},$$

故 (3.4) 成立.

注意到 K 充分小当且仅当

$$\sup_{x\in(0,a)} \varphi_1(x), \sup_{x\in(0,a)} \psi_1(x)$$

及 δ 充分小. 但是

$$\sup_{x\in(0,a)} \varphi_1(x) = \frac{a^{(2-\alpha)}(1-\alpha)^{(1-\alpha)}}{(2-\alpha)^3}, \quad \sup_{x\in(0,a)} \psi_1(x) = \frac{a^{(2-\beta)}(1-\beta)^{(1-\beta)}}{(2-\beta)^3}.$$

故, 选择 a 和 δ 充分小, 则 K 充分小.

设

$$\overline{u}(x,t) = b_1(\delta + \varphi_1(x)), \overline{v}(x,t) = b_2(\delta + \psi_1(x)),$$

则有

$$\overline{u}_t \ge (x^\alpha \overline{u}_x)_x + \int_0^a \overline{v}^n \mathrm{e}^{m\overline{u}} \mathrm{d}x, \ \ \overline{v}_t \ge (x^\beta \overline{v}_x)_x + \int_0^a \overline{u}^q \mathrm{e}^{p\overline{v}} \mathrm{d}x.$$

由引理 2.2, 得到如果 $u_0(x) \le b_1(\delta + \varphi_1(x))$ 且 $v_0(x) \le b_2(\delta + \psi_1(x))$, 则 $(u,v) \le (\overline{u}, \overline{v})$. 因此, (u,v) 整体存在.

定理 3.2 假定下列条件之一成立:

(i) $m > 0$; (ii) $q > 0$; (iii) $m = q = 0$ 且 $np > 1$.

则对充分大的初值 (u_0, v_0), 问题 (1.1) 的解在有限时刻爆破.

证明 情形 (i) $m > 0$. 若 $u_0(x), v_0(x) > 0$, 对所有的 $x \in [0,a]$, $T > 0$ 为 (u,v) 的极大存在时间, 则由极值原理对所有的 $(x,t) \in (0,a) \times (0,T]$ 有 $u(x,t), v(x,t) > 0$, 因此, 对任意给定的子区间 $(0,a') \subset (0,a)$, 存在常数 $k > 0$, 使得当 $(0,a') \times [0,T)$ 时, $u(x,t), v(x,t) \ge k$. 从而,

$$u_t - (x^\alpha u_x)_x \ge \int_0^{a'} v^n \mathrm{e}^{mu} \mathrm{d}x \ge k^n \int_0^{a'} \mathrm{e}^{mu} \mathrm{d}x, \ \ (x,t) \in (0,a') \times (0,T)$$

下面给出如下辅助问题

$$\begin{cases} w_t - (x^\alpha w_x)_x = \mu \int_0^{a'} \mathrm{e}^{mw(x,t)} \mathrm{d}x, \ (x,t) \in (0,a') \times (0,T), \\ w(0,t) = w(a,t) = 0, \ t \in (0,T), \\ w(x,0) = w_0(x), \ x \in [0,a']. \end{cases} \tag{3.5}$$

其中 $\mu > 0$ 为一个常数. 问题 (3.5) 的解对适当的 $w_0(x)$ (参见文献 [14]) 在有限时刻 T_w 爆破. 在问题 (3.5) 中取 $\mu = k^n$, $w_0(x) = u_0(x)$, 由极值原理可以得到对任意的 $(x,t) \in (0,a') \times (0,T)$, $u(x,t) \ge w(x,t)$, 故问题 (1.1) 的解 (u,v) 中的函数 u 当初值 $u_0(x)$ 充分大时在有限时刻爆破, 从而问题 (1.1) 的解 (u,v) 也在有限时刻爆破.

情形 (ii) $p>0$. 类似的, 我们可以证明问题 (1.1) 的解 (u,v) 当初值 $v_0(x)$ 充分大时也在有限时刻爆破.

情形 (iii) 首先, 考虑下述特征值问题

$$\begin{cases} -(x^\alpha\varphi'(x))'=\lambda\varphi(x),\ x\in(0,a), \\ \varphi(0)=\varphi(a)=0, \end{cases} \tag{3.6}$$

$$\begin{cases} -(x^\beta\psi'(x))'=\mu\psi(x),\ x\in(0,a), \\ \psi(0)=\psi(a)=0. \end{cases} \tag{3.7}$$

利用变换 $\varphi(x)=x^{\frac{1-\alpha}{2}}y(x)$, 微分方程 (3.6) 变为

$$x^2y''(x)+xy'(x)-\frac{(1-\alpha)^2}{4}y(x)+\lambda x^{2-\alpha}y(x)=0,\ \ x\in(0,a).$$

再由变换 $y(x)=\phi(z), x=z^{\frac{2}{2-\alpha}}$, (3.6) 变为

$$\begin{cases} z^2\phi''(z)+z\phi'(z)+\left[\frac{4\lambda z^2}{(2-\alpha)^2}-\frac{(1-\alpha)^2}{(2-\alpha)^2}\right]\phi(z)=0,\ x\in(0,a), \\ \phi(0)=\phi(b)=0, \end{cases} \tag{3.8}$$

其中 $b=a^{\frac{2-\alpha}{2}}$. (3.8) 是一个 Bessel 方程, 其解如下

$$\phi(z)=AJ_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda}}{2-\alpha}z\right)+BJ_{-\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda}}{2-\alpha}z\right), \tag{3.9}$$

其中 A 和 B 为任意常数, 而且 $J_{\frac{1-\alpha}{2-\alpha}}$ 和 $J_{-\frac{1-\alpha}{2-\alpha}}$ 分别代表第一类 $\frac{1-\alpha}{2-\alpha}$ 及 $-\frac{1-\alpha}{2-\alpha}$ 阶的 Bessel 函数. 令 λ_1 为

$$J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda}}{2-\alpha}b\right)=0$$

的第一根. 由文献 [15] 的 29 及 75 页可知其正性. 显然 λ_1 是问题 (3.6) 的第一特征值; 同时我们可以得到相对应的特征函数

$$\varphi_2(x)=k_1x^{\frac{1-\alpha}{2}}J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda_1}}{2-\alpha}x^{\frac{1-\alpha}{2}}\right), \tag{3.10}$$

当 $x\in(0,a)$ 时它是正的, 其中 $k_1>0$ 任意. 故可选择 $\varphi_2(x)>0$ 使得 $\max_{x\in[0,a]}\varphi_2(x)=1$.

类似的可以考虑特征值问题 (3.7). 用相同的办法, 令 μ_1 是

$$J_{\frac{1-\beta}{2-\beta}}\left(\frac{2\sqrt{\mu}}{2-\beta}b\right)=0$$

的第一根. 由献 [15] 的 29 及 75 页可知其正性. 显然 μ_1 是问题 (3.7) 的第一特征值; 同时可以得到相对应的特征函数,

$$\psi_2(x)=k_2x^{\frac{1-\beta}{2}}J_{\frac{1-\beta}{2-\beta}}\left(\frac{2\sqrt{\mu_1}}{2-\beta}x^{\frac{1-\beta}{2}}\right), \tag{3.11}$$

当 $x \in (0,a)$ 时它是正的, 其中 $k_2 > 0$ 任意. 故可选择 $\psi_2(x) > 0$ 使得 $\max_{x\in[0,a]} \psi_2(x) = 1$.

假设 $m = q = 0$ 且 $np > 1$. 则存在常数 $n_1, m_1 \geq 1$ 使得 $\frac{1}{p} < \frac{n_1}{m_1} < n$. 设

$$\gamma = \min\{m_1 n - n_1 + 1,\ n_1 p - m_1 + 1\},\ \ c_0 = \max\left\{\max_{x\in[0,a]} \varphi_2^{n_1}(x), \max_{x\in[0,a]} \psi_2^{m_1}(x)\right\},$$

故 $\gamma > 1, c_0 > 0$. 令 $s(t)$ 为下述问题的唯一解

$$\begin{cases} s'(t) = -(\lambda+\mu)s(t) + s^{\gamma}(t)\min\{\frac{1}{n_1 c_0}\int_0^a \psi_2^{m_1 n}(x)\mathrm{d}x, \\ \qquad \frac{1}{m_1 c_0}\int_0^a \varphi_2^{n_1 p}(x)\mathrm{d}x\},\ t > 0, \\ s(0) = s_0 > 0. \end{cases} \tag{3.12}$$

显然当 s_0 充分大时 $s(t)$ 在有限时刻 $T(s_0)$ 爆破. 令

$$\underline{u}(x,t) = s^{n_1}(t)\varphi_2^{n_1}(x), \underline{v}(x,t) = s^{m_1}(t)\psi_2^{m_1}(x), \quad (x,t) \in [0,a]\times[0,T(s_0)).$$

我们将证明 $\underline{u}(x,t), \underline{v}(x,t)$ 是问题 (1.1) 的上解. 直接计算

$$\begin{aligned} &(x^{\alpha}\underline{u}_x)_x + \int_0^a \underline{v}^n \mathrm{d}x \\ &= s^{n_1}(t)[x^{\alpha}(\varphi_2^{n_1}(x))_x]_x + \int_0^a s^{m_1 n}(t)\psi_2^{m_1 n}(x)\mathrm{d}x \\ &= s^{n_1}(t)[n_1\varphi_2^{n_1-1}(x)(x^{\alpha}\varphi_2'(x))' + x^{\alpha}n_1(n_1-1)(\varphi_2'(x))^2\varphi_2^{n_1-2}(x)] + s^{m_1 n}(t)\int_0^a \psi_2^{m_1 n}(x)\mathrm{d}x \\ &\geq n_1 s^{n_1-1}(t)\varphi_2^{n_1}(x)[-\lambda s(t) + s^{m_1 n - n_1 + 1}(t)(\frac{1}{n_1 c_0}\int_0^a \psi_2^{m_1 n}(x)\mathrm{d}x)] \\ &\geq n_1 s^{n_1-1}(t)\varphi_2^{n_1}(x)s'(t) \\ &= \underline{u}_t. \end{aligned}$$

同样 $(x^{\beta}\underline{v}_x)_x + \int_0^a \underline{u}^p \mathrm{d}x \geq \underline{v}_t$. 注意到

$$\underline{u}(0,t) = \underline{u}(a,t) = \underline{v}(0,t) = \underline{v}(a,t) = 0,\ \forall t \in [0,T(s_0)),$$

如果对任意 $x \in (0,a)$, 有

$$\underline{u}(x,0) = s_0^{n_1}\varphi_2^{n_1}(x) < u_0(x), \underline{v}(x,0) = s_0^{m_1}\psi_2^{m_1}(x) < v_0(x),$$

则 $(\underline{u}, \underline{v})$ 是问题 (1.1) 的上解. 因此, 由比较原理, 当初值充分大时问题 (1.1) 的解在有限时刻爆破.

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Blow-up profiles for a degenerate and singular nonlinear parabolic system with nonlocal source

Wang Shengqing [1], He Wansheng [2], Peng Congming [2,3]

(1. School of Mathematics and Information Science, Longnan Teachers College, Longnan 742500, China;

2. School of Mathematics and Statistics, Tianshui Normal University, Tianshui 741000, China;

3. School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000,China)

Abstract: A new degenerate and singular parabolic system with power functions and exponential functions is investigated in this paper. The existence and uniqueness of local solution are proved by using regularization method,moreover the sufficient conditions for the solution that exists globally or blows up in finite time are obtained by method of subsolutions and supersolutions.

Key words: degenerate, global existence, blow up

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